Chapter 11 Material

The solutions to problems from Chapter 11 are presented. Note that the solutions to Problems 11.10 and 11.12 go beyond material covered in the chapter: Problem 11.10 concerns the problem of minimizing cost when the organization has a variety of cost-independent sources for what they sell (in other words, it is a *production allocation* problem); this sort of problem is discussed at length at the end of this chapter, and it comes up again in the *Student's Guide* mterial for Chapter 15, where it connects to the efficiency of competitive-market equilibria. Problem 11.12 ultimately introduces the use of linear programming to solve cost-minimization problems for linear-activity technologies.

11.1 The "bill of materials" for each widget is 10 units of labor, 12 units of material 1, 16 units of material 2, 3 units of electricity, and 5 units of machine-tool time. Given prices, the cost of manufacturing one widget is

$$10 \times \$15 + 12 \times \$1 + 16 \times \$2 + 3 \times \$0.20 + 5 \times \$20 = \$294.60.$$

So the total-cost function for this firm is $\mathrm{TC}(x) = 294.6x$.

11.2 For the graphical approach, see Figure S11.1. The \$40 iso-cost line is supplied and then translated until it is tangent to the 24-unit isoquant. This happens, roughly, at the point $(5.9, 15)$, and so TC(24) is approximately

$$5.9 \times \$4 + 15 \times \$1 = \$38.60.$$

If we do this with calculus, we know that to produce 24 units will take strictly positive amounts of each input, so cost-minimization requires that

$$\frac{r_1}{\mathrm{MPP}_1} = \frac{r_2}{\mathrm{MPP}_2} \quad \text{or} \quad \frac{4}{2y_1^{-1/2}y_2^{1/3}} = \frac{1}{(4/3)y_1^{1/2}y_2^{-2/3}} \quad \text{or} \quad \frac{8y_1}{x} = \frac{3y_2}{x}.$$

To get 24 units, we need $4y_1^{1/2}y_2^{1/3} = 24$, and since $8y_1 = 3y_2$, we can replace y_2 with $(8/3)y_1$, so we have

$$4y_1^{1/2}\left(\frac{8y_2}{3}\right)^{1/3} = 24 \quad \text{or} \quad y_1^{5/6} = 24 \times \frac{1}{4} \times \left(\frac{3}{8}\right)^{1/3},$$

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which gives $y_1 = 5.8$ (roughly) and so $y_2 = (8/3) \times 5.8 = 15.465$. The corresponding total cost is

$$5.8 \times \$4 + 15.465 \times \$1 = \$38.664.$$

So our graphical approach wasn't that far off.

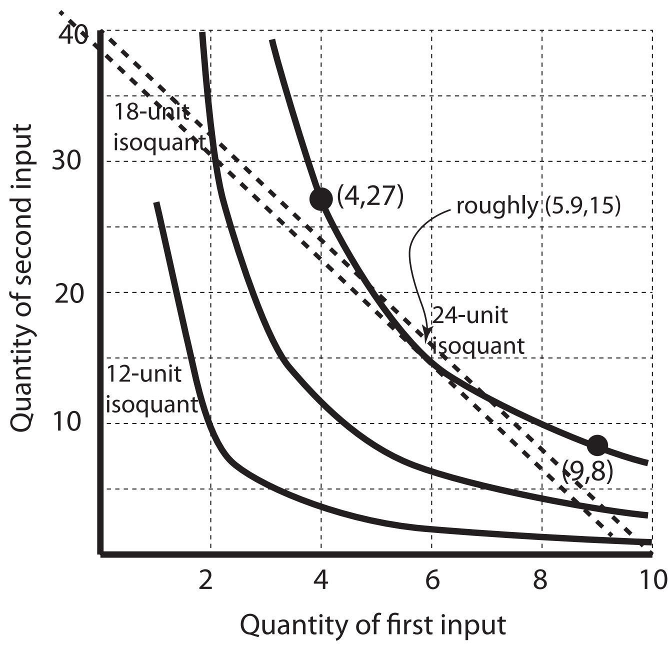


Figure S11.1. Problem 11.2.

(And, in fact, we could have done better with the graphical approach by engaging in a little bit of numerical checking: The point of tangency was read off the graph as (5.9, 15). But is this really enough to produce 24 units? Evaluate $4 \times 5.9^{1/2} \times 15^{1/3}$, and you get 23.96, which isn't far from 24, but is a bit short. Fixing the quantity of x_2 at 15, the required amount of x_1 is $[24/(4 \times 15^{1/3})]^2 = 5.9189$, and the cost of (5.9189, 15) is $5.9189 \times \$4 + 15 \times \$1 = 38.676$, just a bit over one cent more than the calculus-derived answer.)

11.3 Another Cobb-Douglas production function and so we know that to produce any amount above zero, we will need strictly positive amounts of each input. This means that at a cost-minimizing production plan,

$$\frac{r_1}{\mathrm{MPP}_1} = \frac{r_2}{\mathrm{MPP}_2} = \frac{r_3}{\mathrm{MPP}_3}.$$

Do the math for this equation (and for the prices that are given), and you get

$$\frac{6 \times 2y_1}{x} = \frac{1 \times 3y_2}{x} = \frac{0.5 \times 6y_3}{x},$$

when (y_1, y_2, y_3) is selected so that x units of output are produced. Hence $12y_1 = 3y_2 = 3y_3$ at the cost-minimizing production plan.

I'll use this to solve part b directly, and let you finish off the answer to part a: To produce x units, we need $10y_1^{1/2}y_2^{1/3}y_3^{1/6} = x$. I'll substitute $4y_1$ for both y_2 and y_3 in this, and we get

$$10y_1^{1/2}(4y_1)^{1/3}(4y_1)^{1/6} = [10 \times 4^{1/3} \times 4^{1/6}]y_1 = 20y_1 = x,$$

or $y_1 = x/20 = 0.05x$. Hence, $y_2 = y_3 = 4y_1 = 0.2x$, which gives a total-cost function

$$\text{TC}(x) = [0.05 \times \$6 + 0.2 \times \$1 + 0.2 \times \$0.50]x = 0.6x.$$

11.4 (a) To solve the problem graphically, take the 100-unit isoquant, which is reproduced as Figure S11.2, and draw on it some iso-cost line to get the slope. I choose to begin with the \$800 iso-cost, given by the two points $(m = 0,\ l = 200)$ and $(m = 400,\ l = 100)$. (You can choose any level of cost and two points giving that cost that you wish.)

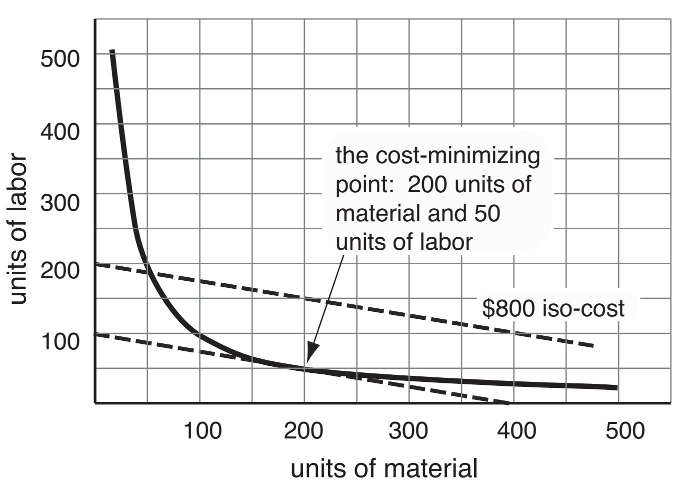


Figure S11.2. Problem 11.4(a): Graphical solution.

Next, we slide this iso-cost line, keeping the slope the same, in or out until it is just tangent to the isoquant. If you, too, picked the \$800 iso-cost, you have to slide inward.

When tangency is achieved, read off the coordinates of the point of tangency ($m = 200$ and $l = 50$) and cost out this set of inputs to get the (cost-minimizing) total cost of producing 100 units:

$$\mathrm{TC}(100) = \$1 \times 200 \quad + \quad \$4 \times 50 = \$400.$$

To solve the problem algebraically, note first that, to produce a strictly positive amount of output, you need strictly positive levels of each input. Therefore, at the cost-minimizing production plan, we must have

$$\frac{r_l}{\mathrm{MPP}_l} = \frac{r_m}{\mathrm{MPP}_m}.$$

The algebra by now should be pretty standard, so I'll leap to the conclusion:

$$2 \times 4l = 2 \times 1m, \quad \text{or} \quad 4l = m.$$

If we wish to produce x units of output and use l units of labor, we will use $4l$ units of material, giving us output $f(l, 4l) = (l)^{1/2}(4l)^{1/2} = 2l$. That is, to produce x units of output in a cost-minimizing fashion, we use $x/2$ units of labor and $2x$ units of material. The cost of this is

$$\mathrm{TC}(x) = \$4 \times \frac{x}{2} \quad + \quad \$1 \times 2x = \$4x.$$

To finish part a, $\mathrm{TC}(100) = \$400$.

(b) If you did part a graphically or with a spreadsheet, at this point you need to invoke my assertion that this technology has constant returns to scale, to conclude that, if 100 units cost \$400, then the marginal cost of production is a *constant* \$4 per unit. If you did part a algebraically, you know the entire total cost function $\mathrm{TC}(x) = 4x$, so you know that $\mathrm{MC}(x) = \$4$. In either case, we go on to equate marginal cost and marginal revenue.

Inverse demand is $P(x) = 12 - (x/2000)$, so total revenue is $\mathrm{TR}(x) = 12x - (x^2/2000)$, and marginal revenue is $\mathrm{MR}(x) = 12 - (2x/2000) = 12 - (x/1000)$. MC = MR where

$$12 - \frac{x}{1000} = 4 \quad \text{or} \quad 8 = \frac{x}{1000} \quad \text{or} \quad x = 8000,$$

at which point the price charged is

$$P(8000) = 12 - \frac{8000}{2000} = \$8.$$

Note that $x = 8000$ implies $l = 4000$ and $m = 16{,}000$.

11.5 This problem is just like Problem 11.4: At a cost-minimizing production plan, the ratios of input price to marginal physical product of the two inputs must be equal, which boils down to

$$3 \times 1 \times m = 6 \times 4 \times l \quad \text{or} \quad m = 8l.$$

To produce x units, we need $m^{1/3}l^{1/6} = x$, and substituting $8l$ for m, this is $(8l)^{1/3}l^{1/6} = 2l^{1/2} = x$, or $l = x^2/4$. This implies $m = 8l = 2x^2$, and so

$$\text{TC}(x) = \$300 + 2x^2 \times \$1 + (x^2/4) \times \$4 = \$300 + \$3x^2.$$

(The \$300 is the fixed cost of the license.) Hence, $\text{MC}(x) = 6x$. Marginal revenue is $\text{MR}(x) = 160 - 4x$, so MC = MR at $160 - 4x = 6x$ or $160 = 10x$ or $x = 16$. This level of production requires 64 units of l and 512 units of m, and gives a price of $160 - 2 \times 16 = \$128$.

11.6 (a) The X is shown on Figure S11.3. My logic is that if the firm has constant returns to scale, then the 20-unit isoquant requires twice the inputs of the 10-unit isoquant. Along the dashed line, the 10-unit isoquant is at the point (14 units of input 1, 10 units of input 2), so the X goes at 28 units of input 1 and 20 units of input 2.

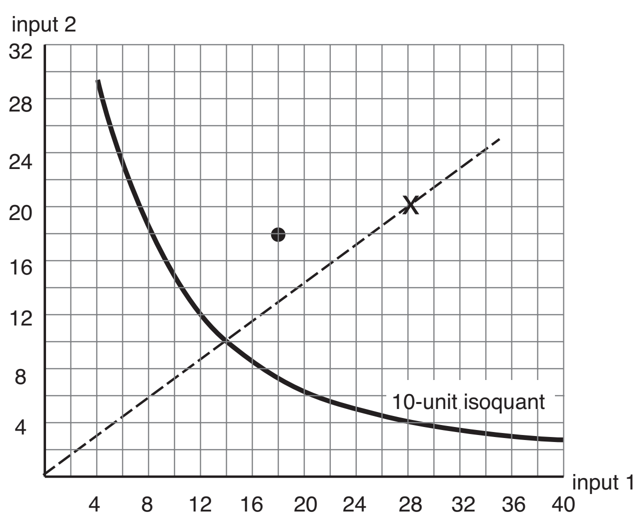


Figure S11.3. Problem 11.6 : An isoquant diagram.

(b) The dot, which is the point $(18, 18)$ is exactly 50% beyond the point $(12, 12)$ which is on the 10-unit isoquant. So with decreasing returns to scale, production at $(18, 18)$ would have to be less than or equal to 15 units. Hence the 12 and 14 unit isoquants could pass through the heavy dot, and the 16 and 18 unit isoquants could not. (Had I asked you about the 15 unit isoquant, the answer would be: It could pass through that point.)

11.7 (a) Returns to scale involve comparisons of the levels of output for input vectors $(y_1, \ldots, y_\ell)$ and $(ay_1, \ldots, ay_\ell)$. Relatively simple algebra tells us that

$$\begin{aligned} f(ay_1, \ldots, ay_\ell) &= K(ay_1)^{\alpha_1}(ay_2)^{\alpha_2} \cdot \ldots \cdot (ay_\ell)^{\alpha_\ell} \\ &= a^{\alpha_1 + \ldots + \alpha_\ell} \times Ky_1^{\alpha_1} \cdot \ldots \cdot y_\ell^{\alpha_\ell} \\ &= a^{\alpha_1 + \ldots + \alpha_\ell} \times f(y_1, \ldots, y_\ell). \end{aligned}$$

If a, the scaling factor, is ≥ 1, then

$$a^{\alpha_1 + \ldots + \alpha_\ell} \text{ is } =, \geq, \leq a \quad \text{as} \quad \alpha_1 + \ldots + \alpha_\ell \text{ is } =, \geq, \leq 1.$$

(b) If the firm wishes to produce 0 output (and assuming, as I do throughout, that all prices are greater than zero), the firm chooses the 0 input vector as least costly. To produce any strictly positive level of output, each of the levels of input must be strictly positive, and so

$$\frac{r_1}{\text{MPP}_1} = \ldots = \frac{r_\ell}{\text{MPP}_\ell}$$

is required. Now for any i,

$$\frac{r_i}{\text{MPP}_i} = \frac{r_i}{\partial f / \partial y_i} = \frac{r_i}{\alpha_i [Ky_1^{\alpha_1} \ldots y_\ell^{\alpha^\ell}] y_i^{-1}} = \frac{r_i y_i}{\alpha_i x},$$

where x is $Ky_1^{\alpha_1} \ldots y_\ell^{\alpha^\ell}$. So, if all these ratios must be equal, we must have

$$\frac{r_1 y_1}{\alpha_1} = \ldots = \frac{r_\ell y_\ell}{\alpha_\ell}, \qquad (\star)$$

which is what we need to show.

(c) From $(\star)$, we have $y_i = (r_1/r_i)(\alpha_i/\alpha_1)\, y_1$ for each i. So in the equation

$$Ky_1^{\alpha_1} \cdot \ldots \cdot y_\ell^{\alpha_\ell} = x,$$

we can substitute for each y_i and get

$$K \times \left[\frac{r_1\alpha_1}{r_1\alpha_1} y_1\right]^{\alpha_1} \times \left[\frac{r_1\alpha_2}{r_2\alpha_1} y_1\right]^{\alpha_2} \times \ldots \times \left[\frac{r_1\alpha_\ell}{r_\ell\alpha_1} y_1\right]^{\alpha_\ell} = x, \text{which is}$$

$$y_1 = \left\{\frac{x}{K} \times \left(\frac{r_1\alpha_1}{r_1\alpha_1}\right)^{\alpha_1} \times \left(\frac{r_2\alpha_1}{r_1\alpha_2}\right)^{\alpha_2} \times \ldots \times \left(\frac{r_\ell\alpha_1}{r_1\alpha_\ell}\right)^{\alpha_\ell}\right\}^{1/(\alpha_1+\ldots+\alpha_\ell)}.$$

Invent constants

$$\beta = \frac{1}{\alpha_1 + \ldots + \alpha_\ell} \quad \text{and} \quad J_1 = \left\{\frac{1}{K} \times \left(\frac{r_1\alpha_1}{r_1\alpha_1}\right)^{\alpha_1} \times \left(\frac{r_2\alpha_1}{r_1\alpha_2}\right)^{\alpha_2} \times \ldots \times \left(\frac{r_\ell\alpha_1}{r_1\alpha_\ell}\right)^{\alpha_\ell}\right\}^{\beta},$$

and this simplifies to $y_1 = J_1 x^\beta$. Moreover, analogous definitions for J_i or, alternatively, setting $J_i = (r_1/r_i)(\alpha_i/\alpha_1)J_1$, gives $y_i = J_i x^\beta$, and so

$$\text{TC}(x) = \left[r_1 J_1 + r_2 J_2 + \ldots r_\ell J_\ell\right] x^\beta.$$

11.8 The firm's cost-minimization problem has $f(y) \geq x$, so the only way that $f(y) = x$ could fail at the cost-minimizing point is if $f(y) > x$. There are two cases to consider:

If $x = 0$ (and if all prices of the inputs are strictly positive, which is assumed throughout; see page 265), then as long as $f(0) = 0$, it is clear that the cost-minimizing production plan is $y = 0$ (any other input vector would have strictly positive cost, while $y = 0$ is adequate to produce $x = 0$ and costs zero). Hence $f(y) = x$ at the solution for $x = 0$ holds.

And, if $x > 0$: Suppose that y is cost-minimizing and satisfies $f(y) > x$. Since $x > 0$ and $f(0) = 0$, we know that y cannot be the zero vector. Now consider the input vector λx for λ a constant very close to but less than 1. As long as $\lambda < 1$ (and, of course, $y \geq 0, y \neq 0$), λy costs strictly less than y (since all prices are strictly positive). But by the continuity of f, if $f(y) > x$, then $f(\lambda y) > x$ for lambda close enough to (but less than) 1. Hence y could not have solved the cost-minimization problem for x; λy for this λ less than but sufficiently close to 1 costs strictly less and still produces $\geq x$ output.

11.9 To produce x units of output, you need

$$x = G\Big(\min\left\{\frac{y_1}{b_1}, \ldots, \frac{y_\ell}{b_\ell}\right\}\Big), \quad \text{which can be rewritten}$$

$$H(x) = \min\left\{\frac{y_1}{b_1}, \ldots, \frac{y_\ell}{b_\ell}\right\}.$$

This implies that $y_1 \geq b_1 H(x), y_2 \geq b_2 H(x)$, and so forth. And there is no gain in output, but increased cost, if any of those weak inequalities is strict. So the cost-minimizing production plan is $y = (y_1, \ldots, y_\ell)$ where each $y_i = b_i H(x)$. This means that total cost is

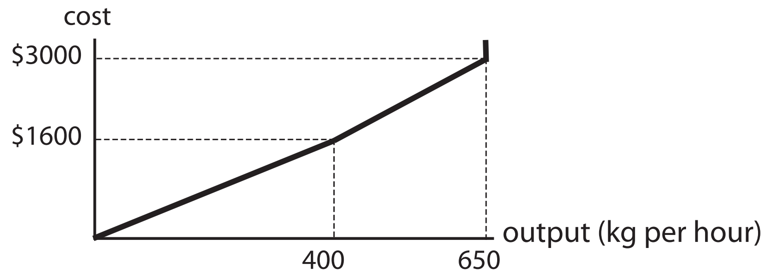
$$\text{TC}(x) = r_1 b_1 H(x) + \ldots + r_\ell b_\ell H(x) = H(x)[r_1 b_1 + \ldots r_\ell b_\ell].$$

Think of it this way: a "balanced input kit" consists of b_1 units of the first input good, b_2 units of the second input good, and so forth. So the cost of a "balanced input kit" is $r_1 b_1 + r_2 b_2 + \ldots + r_\ell b_\ell$. And, to produce x units of output, you need $H(x)$ balanced input kits.

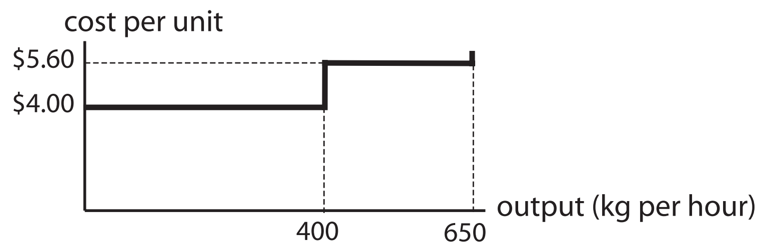
11.10 The solution to Problem 11.10 follows solutions of Problems 11.11 and 11.12, as a broader discussion of the issues this problem raises merits discussion.

11.11 You can make up to 400 kg of final output in the hydration–distillation process at a cost of \$4.00 per kilogram. Why 400 and \$4? Because you can process up to 1000 kg of input, and each kilogram of input provides 0.4 kg of output; moreover, each kilogram of input costs \$1.00 for raw materials and \$20 $\times$ 0.03 = \$0.60 in labor costs, or \$1.60 total, so the cost per kilogram of output is 1.6/0.4 = \$4. And you can make up to 250 kg of final output in the catalytic process at a cost of \$5.60 per kilogram of output: You can process up to 500 kgs of input, getting .5 kgs of output for every kilogram of input; the cost per kilogram of input is \$1.00 for raw materials and \$20 $\times$ 0.09 = \$1.80 for labor, or \$2.80 per kilogram of input total, or \$5.60 per kilogram of output total.

What are your total costs? For up to 400 kg of output, you should use the hydration–distillation process only; your total cost rises linearly at a rate of \$4 per kilogram. At 400 kg of output, you have to shift to the more expensive catalytic process, since you have no more capacity for hydration–distillation; your costs rise at a rate of \$5.60 per kilogram. This works up to an additional 250 kg of output, at which point you can make no more. So we get a total cost function as shown in Figure S11.4(a): linear up to 400 kg, then kinked and linear with a bigger slope up to 650 kg, then vertical, meaning you can get no more output. Figure S11.4(b) graphs the corresponding marginal cost function. Note the staircase character of marginal cost.



(a) Total cost function



(b) Marginal cost function

Figure S11.4. Problem 11.10: Total and marginal cost functions.

11.12 This is an extremely hard problem. In particular, the two technologies are not cost independent, since each uses up shared, scarce inputs to production; namely, cheap and somewhat cheap labor time. If you got close to the answer, you are doing great.

As long as the marginal cost of using additional labor is 0, which is up to where the firm is using 18 labor hours per factory hour, the catalytic process is cheaper per unit output; it has cost of $2.00 per kilogram of output, while the hydration–distillation process has a cost of $2.50 per kilogram. The 18 labor hours stretch to processing 200 kg of raw material in the catalytic process, or 100 kg of output. So, at up to 100 kg of output per hour, we use the catalytic process only, with a marginal cost of $2.00 per kilogram of output.

Once we move beyond 100 kg of output, we have a choice. We can continue to process 200 kg of raw material using the catalytic process and produce additional output using the hydration–distillation process. This, as we computed in Problem 11.10, has a marginal cost of $4.00 per kilogram of output. Alternatively, we can decrease the amount of of catalysis we use, since this uses labor time faster than hydration–distillation, getting back more output with hydration–distillation. How does this work? If we drop 1 kg of output using catalysis, this frees up 0.18 labor hours. With those 0.18 labor hours, we can process 6 kg of input using hydration–distillation, getting 2.4 kg of output. Therefore, the net impact on output is an additional 1.4 kg. The effect on costs is this: We save $2 on the raw material inputs to catalysis, but we spend an extra $6 on raw materials for hydration–distillation, or a net spending of \$4. So, we can get 1.4 kg of final output on the margin at a cost of \$4 on the margin, for a cost of \$2.86 (approximately) per kilogram of final output. This is better than the marginal cost of \$4.00 per kilogram if we increase labor hours, so this is what we do.

This substitution works until we are use all our "free" labor hours on hydration–distillation. The 18 labor hours per factory hour can be used to process 600 kg of input to hydration–distillation, or 240 kg of output. So from 100 kg to 240 kg of output, the marginal cost is \$2.86 per kilogram of output (approximately).

Once we get to 240 kg of output, there is nothing to do but to start employing labor at \$20 per hour. Now, the least marginal cost involves using hydration–distillation, with a marginal cost of \$4 per kilogram of final output. This is good until we process 1000 kg of input for 400 kg of output using hydration–distillation. Note that this employs 30 hours of labor per factory hour, so we have not reached the point at which the price of labor goes up.

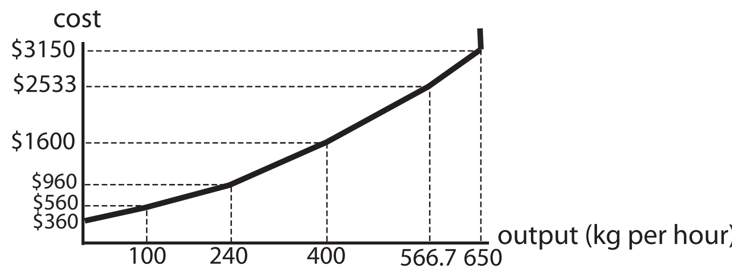
Above 400 kg of output, we have to go back to using catalysis. Now, the marginal cost per kilogram of output is \$5.60 (see Problem 11.11), up to the point where labor usage is 60 hours. Since 30 hours of labor go to hydration–distillation, we have 30 hours that can be used at catalysis, which gives 333.33 kg of input to catalysis, or 166.67 kg of output from catalysis. That is, the marginal cost stays at \$5.60 up to total production of 566.67 kg of output.

Beyond this level, we have to employ premium labor, at \$30 per hour. We continue to want to run as much hydration–distillation as we can; a higher marginal labor cost only reinforces this, so we continue to run a full 1000 kg of input, for 400 kg of output and 30 hours of labor, through hydration distillation. To move beyond 566.67 kgs of output, we have to increase catalysis. The marginal cost of a unit of input to catalysis is \$1 for materials plus $\$30 \times 0.09 = \$2.70$ labor costs, or \$3.70 total, or \$7.40 per unit output. So, up to the overall capacity of 650 kg of output, marginal costs rise to \$7.40 per unit.

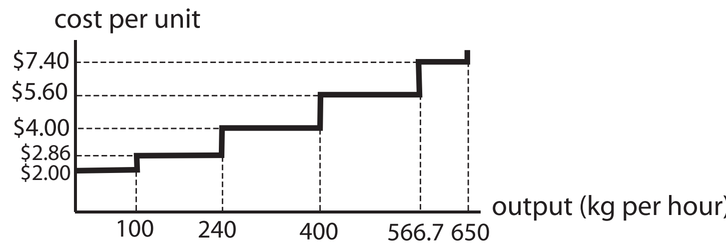
Putting all this together, we get the total and marginal cost functions shown in Figure S11.5. Note the fixed cost for those first 18 labor hours.

Once again, we have a total cost curve that is composed of a number of linear segments (with increasing slope) and a marginal cost curve that looks like a staircase.

Few readers will be able to reason their way through this problem, as we just did. However, if you know how to do linear programming, you can



(a) Total cost function



(b) Marginal cost function

Figure S11.5. Problem 11.12: Total and marginal cost curves.

formulate the computation of total cost as a linear programming problem. Six variables are needed:

- x_0: the hourly amount of kilograms of input per hour, processed using catalysis and fixed-cost labor.
- x_1: the hourly amount of kilograms of input per hour, processed using catalysis and $20 per hour (marginal) labor.
- x_2: the hourly amount of kilograms of input per hour, processed using catalysis and $30 per hour (marginal) labor.
- y_0: the hourly amount of kilograms of input per hour, processed using hydration–distillation and fixed-cost labor.
- y_1: the hourly amount of kilograms of input per hour, processed using hydration–distillation and $20 per hour (marginal) labor.
- y_2: the hourly amount of kilograms of input per hour, processed using hydration–distillation and $30 per hour (marginal) labor.

The objective function is to minimize hourly costs:

minimize $x_0+x_1+x_2+y_0+y_1+y_2+20(0.09x_1+0.03y_1)+30(0.09x_2+0.03y_2)+18\times 20$,

which you should recognize as the costs of the raw material, plus labor costs (where the fixed 18×20 at the end is the fixed cost of the first 18 hours of labor), subject to the following constraints:

- The amount output should equal or exceed the total amount desired: $0.5x_0 + 0.5x_1 + 0.5x_2 + 0.4y_0 + 0.4y_1 + 0.4y_2 \geq C$, where C is a constant for the amount of output being sought.[1]
- The total amount of catalysis input is bounded by 500: $x_0 + x_1 + x_2 \leq 500$.
- The total amount of hydration–distillation input is bounded by 1000: $y_0 + y_1 + y_2 \leq 1000$.
- The amount of fixed-cost labor is bounded above by 18 hours: $0.09x_0 + 0.03y_0 \leq 18$.
- The amount of \$20 labor is bounded above by 42 hours (above the 18 hours of fixed cost labor): $0.09x_1 + 0.03y_1 \leq 42$.
- All the variables are nonnegative: $x_0, x_1, x_2, y_0, y_1, y_2 \geq 0$.

One final note: The "shape" of the total and marginal cost functions (the first composed of a number of linear segments with increasing slope and the other a staircase) is typical of cost functions computed from linear programming formulations. The jumps in marginal cost, corresponding to changes in the slope of total cost, represent basis changes in the optimal solution, as we increase the amount that has to be produced. If you do run this linear program, note that the shadow price on the output constraint, as a function of C, is precisely the marginal-cost function.

Allocation of production/procurement among cost-independent sources

Problem 11.10 is a specific example of the following more general allocation problem: A firm wishes to produce or procure X units of some product, either for resale or for use in its subsequent production processes. The firm has a number of sources that can be used; there are n such sources, and for the ith source ($i = 1, \ldots, n$), the total cost of producing or procuring x_i units from that source is $\mathrm{TC}_i(x_i)$. Source i could be a production facility within the firm, in which case TC_i represents the least-cost way of producing x_i units at that source, and it could represent a B2B transaction with another

[1] Why use a $\geq$ instead of $=$? In fact, I know (how?) this constraint binds at the solution, so it does not matter. But, if this is not true (if the cheapest way to get C units of output is to overproduce), why not do that? (One reason might be that it is costly to dispose of unsold output, but the problem says nothing about that.)

firm: If, say, source i represents purchase from another firm at a fixed price of r_i per unit, then $\mathrm{TC}_i(x_i)$ would be $r_i x_i$.

In dealing with problems of this sort, we always assume that the total costs of production/procurement by the different means are *cost-independent*, meaning that $\mathrm{TC}_i(x_i)$ is not affected by the values of x_j for different j.

For the current discussion, we further assume that TC_i is a differentiable function whose derivative, MC_i (the marginal cost of production/procurement from source i) is nondecreasing. And we assume that $\mathrm{TC}_i(0) = 0$ for all i or, at least, if there are fixed costs associated with source i, they cannot be avoided by setting $x_i = 0$. (In later chapters, we'll relax this assumption somewhat.)

The problem for the firm, then is to

$$\text{Minimize } \mathrm{TC}_1(x_1) + \ldots + \mathrm{TC}_n(x_n), \text{ subject to } x_1 + \ldots + x_n \geq X, \text{ all } x_i \geq 0.$$

The "rule" for finding a solution to this problem is:

The allocation-of-production/procurement plan $(x_1, \ldots, x_n)$ solves this problem if and only if the plan satisfies the constraints of the problem (all the x_i are nonnegative and they sum to X) and, for any $x_i > 0$ and any other j, $MC_i(x_i) \leq MC_j(x_j)$.

So, in particular, if $x_i > 0$ and $x_j > 0$, then $\mathrm{MC}_i(x_i) = \mathrm{MC}_j(x_j)$.

This rule is very intuitive, if you think about it. We've assumed away any fixed costs, so total cost is just the sum of the marginal costs of the different units. If $X = 1$, this unit should be produced/procured from whichever source has the lowest initial marginal cost. Moving to $X = 2$, we look at the marginal costs of all the sources, noting that the marginal cost of the facility used for the first unit may have risen, and go with the smallest marginal cost. And so forth: We keep going to the source with the lowest immediate marginal cost, so any source that we have used will be "tied" for lowest marginal cost of the next unit. The scare quotes around "tied" are there because, if we think of this as a discrete problem (X is an integer and we produce/procure in integer units) "virtually tied" would be better; since we'll be using calculus techniques and continuous variables for the x_i, exactly "tied" is the right rule to apply.

Problem 11.10 provides an example. There are three sources ($n = 3$), with $\mathrm{TC}_1(x_1) = x_1^2/1000 + 3x_1$, $\mathrm{TC}_2(x_2) = x_2^2/2000 + x_2$, and $\mathrm{TC}_3(x_3) = 6x_3$. Hence we get the marginal-cost functions $\mathrm{MC}_1(x_1) = x_1/500 + 3$, $\mathrm{MC}_2(x_2) = x_2/1000 + 1$, and $\mathrm{MC}_3(x_3) = 6$.

Part a of the problem asks what is the cheapest way to get 5000 and then 10,000 units.

We can work our way to the solution as follows. Sources 1 and 2 have rising marginal costs, while the marginal cost of source 3 is the constant 6. Until the marginal costs of sources 1 and 2 hit 6, we'll be using them; once their marginal costs hit 6, we won't use them further but just use source 3. When does this happen?

$$\mathrm{MC}_1(x_1) = \frac{x_1}{500} + 3 = 6 \text{ at } x_1 = 500 \times (6 - 3) = 1500 \text{ units, and}$$

$$\mathrm{MC}_2(x_2) = \frac{x_2}{1000} + 1 = 6 \text{ at } x_2 = 5000.$$

So once total production X rises above 6500, the cost-minimizing plan is to set $x_1 = 1500$, $x_2 = 5000$, and $x_3 = X - 6500$.

That takes care of $X = 10{,}000$, but what about $X = 5000$. We know that, in this case, source 3 won't be used. And the rule says that $\mathrm{MC}_1(x_1) = \mathrm{MC}_2(x_2)$ (if both x_1 and x_2 are strictly positive), so we have two equations in two unknowns:

$$\frac{x_1}{500} + 3 = \frac{x_2}{1000} + 1 \quad \text{and} \quad x_1 + x_2 = 5000. \tag{$\star$}$$

Multiply through the first equation by 1000 and you get $2x_1 + 3000 = x_2 + 1000$, or $2x_1 + 2000 = x_2$ [Question: What does 2000 units represent?]; so we can replace x_2 in the second equation and find

$$x_1 + 2x_1 + 2000 = 5000 \quad \text{or} \quad 3x_1 = 3000 \quad \text{or} \quad x_1 = 1000,$$

and so, of course, $x_2 = 4000$. That's the answer.

Before going on to part b, try one more value: Suppose the firm wanted to procure/produce 1500 units in total.

This is less than 6500, so we know that source 3 won't be used. So hypothesize that sources 1 and 2 will both be used, and the math is the same except that the second equation in ($\star$) is $x_1 + x_2 = 1500$. But now, when we replace x_2 with $2x_1 + 2000$, we get

$$x_1 + 2x_1 + 2000 = 1500 \quad \text{or} \quad 3x_1 = -500.$$

Oops!

This has to do with that 2000 figure. If $X < 2000$, you will by these methods get a negative value of x_1. So the question *What does 2000 units represent?* probably has a good answer. And it does:

The marginal-cost function of source 2, $\mathrm{MC}_2(x_2) = x_2/1000 + 1$, is less than 3 up to $x_2 = 2000$. The marginal cost at source 1, $\mathrm{MC}_1(x_1) = x_1/500 + 3$, has value 3 at $x_1 = 0$ and rises thereafter. Hence, *the cheapest way to source up to 2000 units is to use source 2 exclusively. Source 1 only begins to come into play when X gets above 2000. (And, as already noted, Source 3 only comes into play at $X = 6500$.)*

Now for part b of the problem. The firm faces demand function $D(p) = 400(16 - p)$, so its inverse-demand function is $P(X) = 16 - X/400$ and its marginal-revenue function is $\mathrm{MR}(X) = 16 - X/200$. We'd be in great shape to answer part b if we knew the marginal-cost function for the entire firm. How do we get it?

We construct it. We know that only source 1 is used for X up to 2000 units. So the marginal-cost function up to 2000 units is the same as the marginal cost function of source 2, or $\mathrm{MC}(X) = X/1000 + 1$. And we know that, beyond 6500 units, source 3 is used on the margin, so $\mathrm{MC}(X) = 6$ for $X > 6500$. This leaves the range of X between 2000 and 6500.

Over this range, the marginal cost moves from 3 to 6. So fix a level of marginal cost q between these values and ask: How many units do we get from sources 1 and 2 if we run them both to the point where their marginal costs are q? We run source 1 to the level x_1 that solves $x_1/500 + 3 = q$, or $x_1 = 500(q - 3)$. And we run source 2 to where $x_2/1000 + 1 = q$, or $x_2 = 1000(q - 1)$. So corresponding to marginal costs between 3 and 6, the "supply" from sources 1 and 2 total

$$X(q) = 500(q - 3) + 1000(q - 1) = 1500q - 2500.$$

This gives X as a function of the marginal cost q—to find the marginal-cost function over this range, we invert this to find $q = X/1500 + 2500/1500 = X/1500 + 5/3$. So

$$\mathrm{MC}(X) = \begin{cases} X/1000 + 1, & \text{for } X \leq 2000, \\ X/1500 + 5/3, & \text{for } 2000 \leq X \leq 6500, \text{ and} \\ 6, & \text{for } X \geq 6500. \end{cases}$$

Where does this hit the marginal-revenue function? A graph will help, and in Figure S11.6, I graph both the marginal-revenue and this piecewise linear marginal-cost function.

The intersection occurs on the middle branch of marginal cost (where both source 1 and 2 are being used). So now we can solve

$$16 - \frac{X}{200} = \frac{X}{1500} + \frac{5}{3} \quad \text{or} \quad 48{,}000 - 15X = 2X + 5000,$$

or $43{,}000 = 17X$, or $X = 43{,}000/1700 = 2529.4$. To find out how much is sourced from source 1, solve $3x_1 + 2000 = 2529.4$ to get $x_1 = 176.47$, with the rest coming from source 2.

Let me leave you with one puzzle, to which I will return in the Chapter 15 Material of this *Online Supplement*: The marginal-cost function depicted in Figure S11.6 is the *horizontal sum* of the three individual-source marginal-cost functions. If you really understand how we constructed the overall marginal-cost function, this will make sense. If it doesn't (yet), see if you can get yourself to the point of this level of understanding.

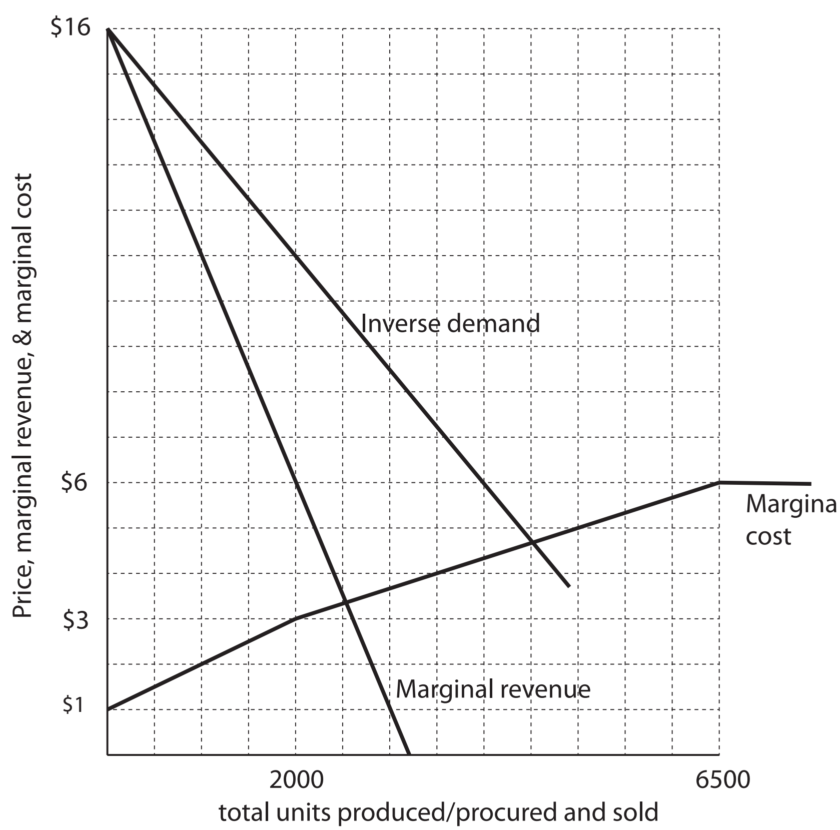


Figure S11.6. Problem 11.10 part b.