Chapter 19 Material

Solutions to the problems from Chapter 19 are provided, followed by a discussion of how insurance companies defeat risk aversion.

19.1 (a) A spreadsheet that does the computations is easy to construct: As a function of the share θ that Jan retains, it computes (1) the utility of the two possible prizes, (2) the expected utility, and (3) the corresponding CE. Figure S19.1(a) (overleaf) shows the results for shares of 0%, 5%, ..., 100%; panel b plots the CE function. (Please reconstruct the spreadsheet that provides panel a.)

(b) You can hunt-and-peck to try to find the percentage share that will maximize CE—per Figure S19.1, it will be between 40 and 50%—or you can employ Solver, asking it to maximize the CE by varying the share. Either way, the answer is around 43.80077%, for a CE of \$2684.03.

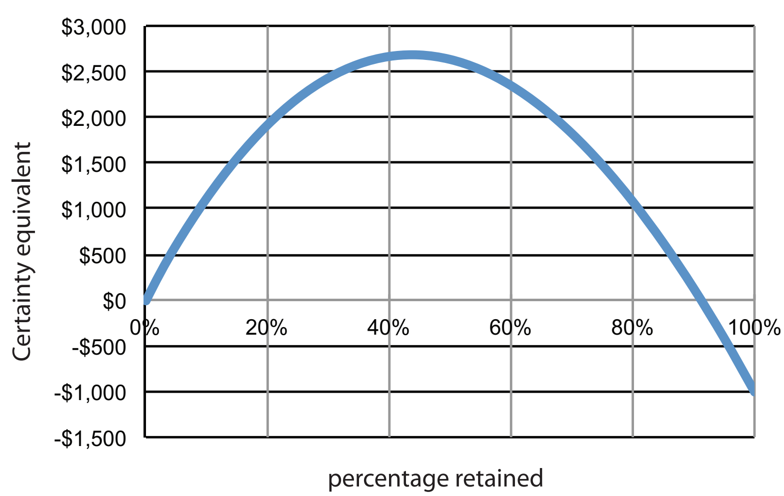
(c) We know from last chapter that the slope at 0% retained will be the EMV. But to verify this numerically, see Table S19.1. I took the spreadsheet and had it compute CE's for 1%, 0.1%, 0.01%, and 0.001% retained, and then divided the resulting CE by the percentage retained. The ratio is very clearly tending to \$12,500, which is the EMV of the gamble.

share retained	good prize	bad prize	utility of good prize	utility of bad prize	EU	CE	ratio of CE to share retained
1%	\$500.000	-\$250.000	-0.9895055	-1.0052889	-0.9973972	\$123.5164217	\$12,351.64
0.10%	\$50.000	-\$25.000	-0.9989456	-1.0005276	-0.9997366	\$12.4851641	\$12,485.16
0.01%	\$5.000	-\$2.500	-0.9998945	-1.0000528	-0.9999736	\$1.2498516	\$12,498.52
0.001%	\$0.500	-\$0.250	-0.9999895	-1.0000053	-0.9999974	\$0.1249985	\$12,499.85

Table S19.1. Problem 19.1(c): The slope of the CE function near 0% retained.

share retained	good prize	bad prize	utility of good prize	utility of bad prize	EU	CE
0%	$0	$0	-1	-1	-1	$0
5%	$2,500	-$1,250	-0.949	-1.027	-0.988	$588
10%	$5,000	-$2,500	-0.900	-1.054	-0.977	$1,102
15%	$7,500	-$3,750	-0.854	-1.082	-0.968	$1,542
20%	$10,000	-$5,000	-0.810	-1.111	-0.961	$1,909
25%	$12,500	-$6,250	-0.768	-1.141	-0.955	$2,204
30%	$15,000	-$7,500	-0.729	-1.171	-0.950	$2,427
35%	$17,500	-$8,750	-0.691	-1.203	-0.947	$2,580
40%	$20,000	-$10,000	-0.656	-1.235	-0.945	$2,665
45%	$22,500	-$11,250	-0.622	-1.268	-0.945	$2,682
50%	$25,000	-$12,500	-0.590	-1.302	-0.946	$2,634
55%	$27,500	-$13,750	-0.560	-1.337	-0.948	$2,522
60%	$30,000	-$15,000	-0.531	-1.372	-0.952	$2,348
65%	$32,500	-$16,250	-0.504	-1.409	-0.956	$2,115
70%	$35,000	-$17,500	-0.478	-1.447	-0.962	$1,824
75%	$37,500	-$18,750	-0.453	-1.485	-0.969	$1,478
80%	$40,000	-$20,000	-0.430	-1.525	-0.977	$1,079
85%	$42,500	-$21,250	-0.408	-1.566	-0.987	$628
90%	$45,000	-$22,500	-0.387	-1.608	-0.997	$129
95%	$47,500	-$23,750	-0.367	-1.651	-1.009	-$416
100%	$50,000	-$25,000	-0.348	-1.695	-1.021	-$1,005

(a) Results from the spreadsheet



(b) Graph of % retained vs. CE

Figure S19.1. Problem 19.1(a): Plotting % retained versus CE.

19.2 (a,b) Panel a of Figure S19.2 provides a basic spreadsheet for this problem. Three parameters are entered: The coefficient of risk aversion of the customer, the percentage of the deal that is desired to be sold to the customer, and the percentage of the EMV that sets the price for the customer. Then, the following are computed: The purchase price of the share sold (EMV $\times$ B4 $\times$ B5), the resulting good and bad outcomes for the customer and their utilities; the EU and the CE. In panel a, you see that trying to sell a 10% share at 95% of the EMV is too "aggressive," the CE for the decision maker is -\$85.70. In panel b, you see the results of searching for the smallest share that this customer will accept (for which he has a positive CE) at 95% of the EMV, which is around 4.20%.

	A	B	C
1	**SPREADSHEET FOR PROBLEM 19.2**		
2			
3	Coefficient of risk aversion	0.0000211	
4	percentage sold	10.00%	
5	percentage of EMV requested	95%	
6			
7	purchase price of share sold	\$1,187.50	
8			utility
9	good outcome	\$3,812.50	-0.9227068
10	bad outcome	-\$3,687.50	-1.0809132
11	EU		-1.00181
12	CE		-\$85.70

(a) Basic spreadsheet

	A	B	C
1	**SPREADSHEET FOR PROBLEM 19.2**		
2			
3	Coefficient of risk aversion	0.0000211	
4	percentage sold	4.20%	
5	percentage of EMV requested	95%	
6			
7	purchase price of share sold	\$498.75	
8			utility
9	good outcome	\$1,601.25	-0.966778
10	bad outcome	-\$1,548.75	-1.0332184
11	EU		-0.9999982
12	CE		\$0.08

(b) Largest acceptable share at 95% of EMV

Figure S19.2. Problem 19.2(a) and (b): Finding the largest acceptable share at 95% of EMV.

(c) Figure S19.3 provides the results. I copied columns B and C several times to work out different scenarios. For the customer whose coefficient of risk aversion is $\lambda = 0.0000211$, to get 98% of the EMV took a share of around 1.68% (columns B and C); for the customer whose coefficient of risk aversion is 0.00001, 98% of EMV took a 3.55% share (columns E and F) and 95% of EMV took an 8.89% share (columns H and I). Note that these are higher acceptable shares for the same percentage of EMV as for the first customer: The coefficient λ is the coefficient of risk *aversion*, and a bigger λ (in the sense that 0.0000211 is bigger than 0.00001) means *more* risk averse.

(d) Figure S19.4 does the analysis for a customer with utility function $\sqrt{x + 50,000}$. The biggest acceptable share at a price of 98% of EMV is 3.555%, while for 95% of EMV, it is 8.888%.

These numbers are "suspiciously" close to the cutoff percentages for the customer with utility function $-e^{-0.00001x}$. The problem asks you to evaluate

	A	B	C	D	E	F	G	H	I
4	percentage sold	1.68%			3.55%			8.89%	
5	percentage of EMV requested	98%			98%			95%	
6									
7	purchase price of share sold	$205.80			$434.88			$1,055.45	
8			utility			utility			utility
9	good outcome	$634.20	-0.9867075		$1,340.13	-0.9866881		$3,388.55	-0.9666822
10	bad outcome	-$625.80	-1.0132919		-$1,322.38	-1.0133116		-$3,277.45	-1.0333175
11	EU		-0.9999997			-0.9999999			-0.9999998
12	CE		$0.01			$0.01			$0.01

Figure S19.3. Problem 19.2(c).

	A	B	C	D	E	F
42	percentage sold	3.555%			8.888%	
44	percentage of EMV requested	98%			95%	
46	purchase price of share sold	$435.49			$1,055.45	
47			utility			utility
48	good outcome	$1,342.01	226.587759		$3,388.55	231.059624
49	bad outcome	-$1,324.24	220.625843		-$3,277.45	216.153996
50						
51	EU		223.606801			223.60681
52	CE		0.00138892			0.00556118

Figure S19.4. Problem 19.2(d).

$-u''/u'$ at the value 0: For the square-root utility function $u'(x) = (1/2)(x + 50{,}000)^{-1/2}$ and $u''(x) = -(1/4)(x + 50{,}000)^{-3/2}$, so $-u''/u'$ evaluated at 0 is

$$\frac{(1/4)(50{,}000)^{-3/2}}{(1/2)(50{,}000)^{-1/2}} = \frac{1}{2}\,\frac{1}{50{,}000} = \frac{1}{100{,}000} = 0.00001.$$

(If you evaluate $-u''/u'$ for $u(x) = -e^{-0.00001x}$, it is 0.00001 at *all* values of x.) This is no coincidence, although you'll need to consult a more advanced book than this one to figure out what is going on.

19.3 We use a modified version of the spreadsheet employed for Problem 19.1. Now, if Jan retains share α, she gets $\$50{,}000\alpha + \$12,500(1 - \alpha)$ with probability 0.7 and $-\$25,000\alpha + \$12,500(1 - \alpha)$ with probability 0.3. The results—for various percentages that she retains, her CE—are tabulated in Table S19.2. This indicates that she wants to retain around 55%. If you use Solver to fine-tune this, the optimal percentage turns out to be around 53.54%.

19.4 The spreadsheet required for this problem is a bit more complex, because the probability of success (and also the price she gets per share) changes with the percentage she retains (and the percentage she sells). I found it easiest to put in a column for "probability of success," which I input

share retained	good prize	bad prize	utility of good prize	utility of bad prize	EU	CE
0%	$12,500	$12,500	-0.76817	-0.76817	-0.76817	$12,500
5%	$14,375	$10,625	-0.73837	-0.79917	-0.75661	$13,219
10%	$16,250	$8,750	-0.70973	-0.83142	-0.74623	$13,873
15%	$18,125	$6,875	-0.6822	-0.86497	-0.73703	$14,461
20%	$20,000	$5,000	-0.65573	-0.89987	-0.72898	$14,982
25%	$21,875	$3,125	-0.6303	-0.93619	-0.72207	$15,433
30%	$23,750	$1,250	-0.60585	-0.97397	-0.71629	$15,814
35%	$25,625	-$625	-0.58235	-1.01327	-0.71163	$16,123
40%	$27,500	-$2,500	-0.55976	-1.05417	-0.70808	$16,360
45%	$29,375	-$4,375	-0.53805	-1.09671	-0.70564	$16,523
50%	$31,250	-$6,250	-0.51717	-1.14097	-0.70431	$16,613
55%	$33,125	-$8,125	-0.49711	-1.18701	-0.70408	$16,628
60%	$35,000	-$10,000	-0.47783	-1.23491	-0.70495	$16,570
65%	$36,875	-$11,875	-0.4593	-1.28475	-0.70693	$16,437
70%	$38,750	-$13,750	-0.44148	-1.33659	-0.71001	$16,231
75%	$40,625	-$15,625	-0.42435	-1.39053	-0.71421	$15,952
80%	$42,500	-$17,500	-0.40789	-1.44665	-0.71952	$15,601
85%	$44,375	-$19,375	-0.39207	-1.50503	-0.72596	$15,178
90%	$46,250	-$21,250	-0.37686	-1.56577	-0.73353	$14,686
95%	$48,125	-$23,125	-0.36224	-1.62895	-0.74226	$14,126
100%	$50,000	-$25,000	-0.34819	-1.69469	-0.75214	$13,499

Table S9.2. Problem 19.3.

as numbers, rather than creating a complex IF(. . .) formula. In any event, the results are tabulated in Table S19.3. As you can see (looking at the numbers in Table S19.3), she wishes to retain 70% of the venture for herself. (The fact that the answer, 70%, is a round number is an artifact of the model. If she retains 70%, the probability of success is 0.65, while if she sells even 70.1%, it falls to 0.60. In real life, this discontinuity would not be present.

19.5 This problem is solved in detail at the start of Chapter 20, so I omit its solution here.

19.6 Because Biff will buy at most one 1% share, and because he has exponential utility, all we need to do is to work out Biff's CE if he combines his gamble with 1% of Jan's gamble, 1% of Joe's, and 1% of Jess's. To do this, we need to know the probabilities of the four outcomes $2,000,500, $1,999,750, $1,800,500, and $1,799,750, for each of the three combinations.

Jan's gamble is independent of Biff's, so the four probabilities are 0.25 each.

Joe's gamble pays off $50,000 with probability 0.6, conditional on Biff's paying off $2 million. So the joint probability that Joe's pays off $50,000 and Biff's pays off $2 million is 0.3. (Use the rule that $P(A, B) = P(A|B) \times P(B)$.) From simple rules of probability, this says that the probability that Joe's gamble

share retained	probability of success	EMV of venture	good prize	bad prize	utility of good prize	utility of bad prize	EU	CE
0%	0.50	12500.00	$11,875	$11,875	-0.77836	-0.77836	-0.77836	$11,875
5%	0.50	12500.00	$13,781	$10,031	-0.74768	-0.80924	-0.77846	$11,869
10%	0.50	12500.00	$15,688	$8,188	-0.7182	-0.84134	-0.77977	$11,789
15%	0.50	12500.00	$17,594	$6,344	-0.68989	-0.87472	-0.7823	$11,636
20%	0.50	12500.00	$19,500	$4,500	-0.66269	-0.90942	-0.78605	$11,409
25%	0.50	12500.00	$21,406	$2,656	-0.63656	-0.94549	-0.79103	$11,110
30%	0.50	12500.00	$23,313	$812	-0.61147	-0.983	-0.79723	$10,740
35%	0.50	12500.00	$25,219	-$1,031	-0.58736	-1.022	-0.80468	$10,299
40%	0.50	12500.00	$27,125	-$2,875	-0.56421	-1.06254	-0.81337	$9,790
45%	0.50	12500.00	$29,031	-$4,719	-0.54196	-1.10469	-0.82333	$9,213
50%	0.60	20000.00	$34,500	-$3,000	-0.4829	-1.06535	-0.71588	$15,841
55%	0.60	20000.00	$36,050	-$5,200	-0.46736	-1.11597	-0.7268	$15,123
60%	0.60	20000.00	$37,600	-$7,400	-0.45232	-1.16899	-0.73899	$14,335
65%	0.60	20000.00	$39,150	-$9,600	-0.43777	-1.22453	-0.75247	$13,478
70%	0.65	23750.00	$41,769	-$10,731	-0.41424	-1.25411	-0.70819	$16,353
75%	0.65	23750.00	$43,141	-$13,109	-0.40242	-1.31865	-0.7231	$15,365
80%	0.65	23750.00	$44,513	-$15,488	-0.39094	-1.38651	-0.73938	$14,310
85%	0.65	23750.00	$45,884	-$17,866	-0.37978	-1.45785	-0.75711	$13,187
90%	0.70	27500.00	$47,613	-$19,888	-0.36618	-1.52139	-0.71275	$16,049
95%	0.70	27500.00	$48,806	-$22,444	-0.35707	-1.60571	-0.73166	$14,807
100%	0.70	27500.00	$50,000	-$25,000	-0.34819	-1.69469	-0.75214	$13,499

Table S19.3. Problem 19.4.

pays off $-\$25{,}000$ and Biff's pays off $2 million is 0.2, the probability that Joe's pays off $50,000 and Biff's pays of $1.8 million is 0.2, and the probability that Joe's pays off $-\$25{,}000$ and Biff's pays off $1.8 million is 0.3.

Similarly, we get the four joint probabilities for Biff's gamble combined with Jess's that are shown in Figure S19.5. Note that Jan's gamble is independent of Biff's original portfolio, Joe's is positively correlated, and Jess's is negatively correlated.

With these probabilities, for each combination, we can compute utilities of the prizes, then expected utilities of the gambles, and then CEs of the gambles. All this is done in the spreadsheet depicted in Figure S19.5

The difference between the CEs of the joint-lottery gambles (where Biff buys a 1% share of the Jan's, then Joe's, then Jess's gamble) and the CE of the "no added risk" gamble is the most Biff would pay. Note that he pays the most, $196, for a 1% share of Jess's gamble, and the least, $51, for a 1% share of Joe's. This is because Joe's gamble is positively correlated with his own, while Jess's gamble is negatively correlated. Note also that the expected value of Jan's, Joe's, and Jess's gambles is (the same) $12,500, so a 1% share has an expected value of $125; for the case of Jan's gamble, which is independent, the 1% share is worth a bit less than its EMV.

	A	B	C	D	E	F
2						
3				outcome	probability	utilities
4	SCRENARIO 0	base outcomes		$2,000,000	0.5	-4.248E-18
5	No added risk			$1,800,000	0.5	-2.32E-16
6						
7		expected utility		-1.181E-16		
8		certainty equivalent		$1,833,750		
9						
10	SCENARIO 1	full outcomes		$2,000,500	0.25	-4.206E-18
11	Independent risk			$1,999,750	0.25	-4.27E-18
12	Jan's gamble			$1,800,500	0.25	-2.296E-16
13				$1,799,750	0.25	-2.331E-16
14						
15		expected utility		-1.17809E-16		
16		certainty equivalent		$1,833,873		
17		difference in CEs		$124		
18						
19	SCENARIO 2	full outcomes		$2,000,500	0.3	-4.206E-18
20	positive correlation			$1,999,750	0.2	-4.27E-18
21	Joe's gamble			$1,800,500	0.2	-2.296E-16
22				$1,799,750	0.3	-2.331E-16
23						
24		expected utility		-1.17979E-16		
25		certainty equivalent		$1,833,801		
26		difference in CEs		$51		
27						
28	SCENARIO 3	full outcomes		$2,000,500	0.2	-4.206E-18
29	Negative correlation			$1,999,750	0.3	-4.27E-18
30	Jess's gamble			$1,800,500	0.3	-2.296E-16
31				$1,799,750	0.2	-2.331E-16
32						
33		expected utility		-1.17638E-16		
34		certainty equivalent		$1,833,946		
35		difference in CEs		$196		

Figure S19.5. Problem 19.6: A spreadsheet to solve the problem.

19.7 (a) This requires, conceptually, a simple application of Solver. If you create the spreadsheet that was used to compute the numbers in Figure 19.1, you then ask Solver to maximize Ringo's CE (or expected utility), subject to holding John, Paul, and George's CEs (or EUs) at their initial levels, varying the sharing rule amounts for John, Paul, and George, and giving Ringo whatever is left over after giving John, Paul, and George their shares.

As noted in the statement of the problem, to get Solver to work on this, I needed to choose the option of automatic scaling. Once that was done, I got

	A	B	C	D	E	F
3			state 1	state 2	state 3	state 4
4		Total	$100,000.00	$200,000.00	$300,000.00	$400,000.00
5						
6		John	$31,725.77	$47,799.90	$65,109.31	$83,249.03
7		Paul	$645.26	$51,600.56	$106,567.78	$164,117.50
8		George	$34,632.01	$47,106.04	$59,378.77	$71,396.83
9		Ringo	$32,996.96	$53,493.50	$68,944.14	$81,236.64
10						
11		probs	0.4	0.3	0.2	0.1
12						
13		John's utility	178.1172967	218.6318798	255.1652602	288.5290766
14		Paul's utility	317.2463735	389.3591652	454.4972844	513.923628
15		George's utility	32.58426347	36.10242924	38.99880344	41.46972868
16		Ringo's utility	-0.718945595	-0.58570734	-0.501856344	-0.443806584
17						
18			EU	CE		
19		John	216.7224423	$46,968.62		
20		Paul	385.9981186	$48,994.55		
21		George	35.81116771	$45,925.66		
22		Ringo	-0.608042368	$49,751.07		

Figure S19.6. Problem 19.7: The answer according to Solver.

back the answers shown in Figure C19.6.

It is perhaps worth noting the general nature of the "answer." Paul is the least risk averse, so his shares show the widest dispersion. There is little difference in George, John, and Ringo, although George seems the most risk averse.

(Optional: I can check that this is the answer, by looking across the Fab Four at the ratios of their marginal utilities in various states. See Figure C19.7, which should be self-explanatory to anyone reading this far into a parenthetical remark marked "optional.")

(b) This is easy. Ringo is risk neutral, so efficient risk sharing means he gets all the risk, and John, Paul, and George get sure things. Since we want the three to have CEs of $50,000 each, we give them sure-thing payments of $50,000 each, and Ringo sucks up what is left: If the overall outcome is $100,000, Ringo has to lay out $50,000. If the overall outcome is $200,000, Ringo takes home $50,000. If the overall outcome is $300,000, Ringo takes home $150,000. And if the overall outcome is $400,000, Ringo makes the big bucks: He gets $250,000. Given the probabilities of the four outcomes, this gives Ringo an EMV (= CE) of $50,000.

	A	B	C	D	E	F	G	H	I	J
1										
2										
3			state 1	state 2	state 3	state 4				
4		Total	$100,000.00	$200,000.00	$300,000.00	$400,000.00				
5										
6		John	$31,725.77	$47,799.90	$65,109.31	$83,249.03				
7		Paul	$645.26	$51,600.56	$106,567.78	$164,117.50				
8		George	$34,632.01	$47,106.04	$59,378.77	$71,396.83				
9		Ringo	$32,996.96	$53,493.50	$68,944.14	$81,236.64				
10										
11		probs	0.4	0.3	0.2	0.1				
12										
13		John's utility	178.1172967	218.6318798	255.1652602	288.5290766				
14		Paul's utility	317.2463735	389.3591652	454.4972844	513.923628				
15		George's utilit	32.58426347	36.10242924	38.99880344	41.46972868				
16		Ringo's utility	-0.718945595	-0.585707342	-0.501856344	-0.443806584				
17										
18			EU	CE						
19		John	216.7224423	$46,968.62						
20		Paul	385.9981186	$48,994.55						
21		George	35.81116771	$45,925.66						
22		Ringo	-0.608042368	$49,751.07						
23								ratio 1:2	ratio 1:3	ratio 1:4
24										
25		John's MUs	0.002807	0.002287	0.001960	0.001733		1.22746013	1.43256868	1.61988241
26		Paul's MUs	0.001576	0.001284	0.001100	0.000973		1.22730848	1.43263193	1.61995115
27		George's MUs	0.000314	0.000256	0.000219	0.000194		1.22762687	1.43252806	1.61982674
28		Ringo's Mus	0.000007	0.000006	0.000005	0.000004		1.22748264	1.4325725	1.61995252
29										

Figure S19.7. Problem 19.7: I know Figure S19.6 is the answer because . . . the ratios of marginal utilities across states are equalized across the Fab Four.

How Do Insurance Companies Defeat Risk Aversion? Risk Spreading or the Law of Averages?

Having introduced risk spreading and sharing as the raison d'être for insurance companies, let me say a few more words about how insurance companies defeat risk aversion.

Think of an insurance company as a body that combines many independent risks and then spreads the risk out among a large number of shareholders. Most popular accounts of the insurance business invoke the law of averages or of large numbers to explain why this arrangement leads to risk neutrality. The idea is that, when insurance companies put together a lot of risks in a portfolio, the good outcomes cancel out the bad and, if the "average" payout is less than the premium paid on the policy, the good outcomes (policies where the premium more than covers any payout) outweigh the bad, where the payout exceeds the premium.

This is *not* the story told in Chapter 19. Chapter 19 says that insurance companies are close to risk neutral because they spread the risk over a large number of individuals, each of whom is therefore nearly risk neutral for his or her share of the risk.

Notwithstanding all those popular accounts, the second story captures the truth. If the first story is right (if the portfolio of many risks does the trick), then insurance companies with a single shareholder, bearing all the risk of the portfolio, would work. If the second story is right (if the explanation for why insurance companies work is risk spread over a large number of individuals), then an insurance company with many small shareholders that insures a single risk would work.

Of course, we already know that the second story works. If you take a single risk and parcel it out in many small shares to many individuals, its value rises to its expected monetary value, because each of the small shares represents a very small amount of risk. This is true even if it is a single risk; if no "law of averages" is at work at all.

(But remember the discussion in the text about correlated risks. For an insurance company whose business is, say, earthquake insurance for homeowners in California, individual policies are very positively correlated; the parceling out of this risk will require a lot of shareholders who don't suffer from earthquakes in California.)

As for the first story, when we sum up the risks in an insurance company's portfolio, it is certainly true that some good outcomes cancel out some bad outcomes. But there is a lot of risk in the sum, more than any single investor would be comfortable taking on. The law of averages (that the good outcomes almost surely outweigh the bad outcomes, assuming the expected value of each risk is positive) applies not to the sum of a collection of risks but to their average. And we are looking at average outcomes only if we take the insurance company's large portfolio and divide it into pieces; that is, if we share the risk.