Chapter 6 Material

Solutions to the problems from Chapter 6 follow.

6.1 If inverse demand is given by $P(x) = 100 - x/100$, then $\mathrm{TR}(x) = 100x - x^2/100$ and $\mathrm{MR}(x) = 100 - x/50$. If $\mathrm{TC}(x) = 200 + 20x + x^2/300$, then $\mathrm{MC}(x) = 20 + x/150$. So MR = MC is

$$100 - \frac{x}{50} = 20 + \frac{x}{150} \quad \text{or} \quad 80 = \frac{4x}{150} \quad \text{or} \quad x = 3000,$$

which gives a profit-maximizing price of 70.

6.2 Inverse demand $P(x) = 20 - x/1000$ means $\mathrm{MR}(x) = 20 - 2x/1000 = 20 - x/500$. Total cost $5000 + 4x$ means $\mathrm{MC}(x) = 4$. So MR = MC is

$$4 = 20 - \frac{x}{500} \quad \text{or} \quad x = 500 \times 16 = 8000,$$

for a profit-maximizing price of 12.

6.3 (a) To set $\mathrm{AC}' = 0$, you first take the derivative of the function $\mathrm{AC}(x) = \mathrm{TC}(x)/x = [10{,}000{,}000 + 200x + x^2/1000]/x = 10{,}000{,}000/x + 200 + x/1000$, which is

$$\mathrm{AC}'(x) = -\frac{10{,}000{,}000}{x^2} + \frac{1}{1000}.$$

Set this equal to zero, and you have the equation $x^2 = 10{,}000{,}000 \times 1000 = 10{,}000{,}000{,}000$; taking square-roots on both sides is $x = 100{,}000$. So 100,000 is efficient scale. And minimum average cost is the value of average cost at this level of production, or

$$\mathrm{AC}(100,000) = \frac{10{,}000{,}000}{100{,}000} + 200 + \frac{100{,}000}{1000} = 100 + 200 + 100 = 400.$$

Or you can set AC = MC, which is the equation

$$\frac{10{,}000{,}000}{x} + 200 + \frac{x}{1000} = 200 + \frac{2x}{1000}.$$

Cancel the 200's on each side, and move $x/1000$ from the left-hand side to the right-hand side, and you get

$$\frac{10{,}000{,}000}{x} = \frac{x}{1000} \quad \text{or} \quad 10{,}000{,}000 \times 1000 = x^2,$$

at which point the algebra is precisely as before.

(b) If x is the level of domestic steel sales and y is the level of export sales, then total revenue is

$$\mathrm{TR}(x, y) = \left(1000 - \frac{x}{250}\right)x + 375y,$$

so that marginal revenue in x (loosely, the additional revenue from one more ton of steel sold domestically and, precisely, the partial derivative of TR(x, y) in x) is $1000 - x/125$, while the marginal revenue in exported steel is 375.

As for costs: The total production quantity is $x + y$, so

$$\mathrm{TC}(x, y) = 10{,}000{,}000 + 200(x + y) + \frac{(x + y)^2}{1000.}$$

Marginal cost in x is the partial derivative of this with respect to x, or

$$200 + \frac{2(x + y)}{1000} = 200 + \frac{x + y}{500}.$$

To be profit-maximizing, the marginal revenue in x must be equal to the marginal cost in x, or

$$200 + \frac{x + y}{500} = 1000 - \frac{x}{125},$$

and similarly for y, which is

$$200 + \frac{x + y}{500} = 375.$$

(It is worth observing that, since TC is a function of $x + y$, the marginal cost in x, or that rate at which total cost increases per unit increase in x, must of course equal the marginal cost in y.) In any case, you can use the second of these conditions to solve for $x + y$, and, since two things equal to the same (third) thing are equal to each other, you can set $375 = 1000 - x/125$ to solve for x. Which is just what happened in the text.

6.4 There are two ways you can solve this. The first, and (I hope) the one that occurs to you first, is to invert the two demand functions to get the two inverse demand functions and then proceed as in the text. To see how this goes, start with

$$x = \frac{45{,}000 - 900P_X + 300P_Y}{7} \quad \text{and} \quad y = \frac{54{,}000 - 900P_Y + 600P_X}{7}$$

to get

$$7x = 45{,}000 - 900P_X + 300P_Y \quad \text{and} \quad 7y = 54{,}000 - 900P_Y + 600P_X.$$

Multiply the first of these by 2 and the second by 3:

$$14x = 90{,}000 - 1800P_X + 600P_Y \quad \text{and} \quad 21y = 162{,}000 - 2700P_Y + 1800P_X.$$

Add them together (to eliminate P_X): $14x + 21y = 252{,}000 - 2100P_Y$, or

$$P_Y = \frac{252{,}000 - 14x - 21y}{2100} = 120 - \frac{x}{150} - \frac{y}{100},$$

which should remind you of half the first display on page 138. And, of course, if you solve for P_X, you will get the other half. From here, just do the five steps found on page 138.

Or, *alternatively*, you can work entirely in prices. I'm going to write $\hat{\pi}(P_X, P_Y)$ for the profit the firm earns as a function of the two prices that it sets; the hat will distinguish this function of prices from $\pi(x, y)$, which is profit expressed as a function of the quantities the firm sells. We have

$$\hat{\pi}(P_X, P_Y) = P_X\left[\frac{45{,}000 - 900P_X + 300P_Y}{7}\right] + P_Y\left[\frac{54{,}000 - 900P_Y + 600P_X}{7}\right]$$

$$-10\left[\frac{45{,}000 - 900P_X + 300P_Y}{7}\right] - 20\left[\frac{54{,}000 - 900P_Y + 600P_X}{7}\right] - 100.$$

This is the revenue from selling X's plus the revenue from selling Y's less the total cost, which is 10 times the number of X's plus 20 times the number of Y's plus 100. Now take the partial derivatives of $\hat{\pi}$ in P_X and P_Y and set those two partials to zero: For the partial with respect to P_X, we get

$$\left[\frac{45{,}000 - 900P_X + 300P_Y}{7}\right] - \left[\frac{900}{7}\right]P_X + \left[\frac{600}{7}\right]P_Y + 10\left[\frac{900}{7}\right] - 20\left[\frac{600}{7}\right].$$

(I'm using the product rule in ways that may be less than transparent here, so if you don't see how I got this as the partial derivative, take the expression for $\hat{\pi}$ and distribute all the multiplications, and then take the derivative.) This has to equal 0, so we can drop all the 7's in the denominators and collect terms, and we get

$$42{,}000 - 1800P_X + 900P_Y = 0. \qquad (\star)$$

And for the partial derivative of $\hat{\pi}$ in P_Y, we get

$$P_X\left[\frac{300}{7}\right] + \left[\frac{54000 - 900P_Y + 600P_X}{7}\right] - \left[\frac{900}{7}\right]P_Y - 10\left[\frac{300}{7}\right] + 20\left[\frac{900}{7}\right].$$

Setting this equal to zero (after multiplying through by 7) and collecting terms gives

$$69000 - 1800P_Y + 900P_X = 0. \qquad (\star)$$

And, if you solve the two starred equations simultaneously, you get $P_X = 56.666\ldots$ and $P_Y = 66.666\ldots$, which is the answer we got in the text.

Neither of these methods is particularly pleasant; both involve too much algebra. But they get the right answer in the end, if you do the math correctly. You'll almost surely never have to do something like this in real life, but if a problem like this ever comes up on an exam, my advice would be to use method #1. I've seen too many students get confused when they start trying to think of marginal profit where the variable being shifted is price. But they do both work.

6.5 This problem is not much more difficult than the two-tablet problem; the only complication is that the total cost function is a bit more complex. Total profit, $\pi(x, y)$, in this case can be written as

$$\left[100 - \frac{x}{100} - \frac{y}{400}\right]x + \left[80 - \frac{y}{50} - \frac{x}{200}\right]y - \left[300 + 20x + 10y + \frac{9x^2 + 6xy + y^2}{1200}\right],$$

where I already began some algebraic manipulation by computing the squared term in the cost function. Skipping some algebraic steps, this simplifies to

$$80x + 70y - \frac{21x^2}{1200} - \frac{25y^2}{1200} - \frac{15xx}{1200} - 300.$$

We must compute the partial derivatives in x and y and set each to 0: This gives

$$\frac{\partial\pi(x,y)}{\partial x} = 80 - \frac{42x}{1200} - \frac{15y}{1200} = 0, \quad \text{and}$$

$$\frac{\partial\pi(x,y)}{\partial y} = 70 - \frac{50y}{1200} - \frac{15x}{1200} = 0.$$

We now have two simultaneous linear equations in two unknowns. And with a bit of work, we get the solution

$$x = 1888 \quad \text{and} \quad xy = 1113.6.$$

6.6 (a) Marginal revenue from Wolverton is $20 - W/1000$, while from Manteca is is $24 - M/1500$. These are equal when

$$20 - \frac{W}{1000} = 24 - \frac{M}{1500}.$$

We also have $W + M = 30{,}000$, and we need to solve these two equations in two unknowns. Write $M = 30{,}000 - W$ and replace M in the displayed equation:

$$20 - \frac{W}{1000} = 24 - \frac{30{,}000 - W}{1500} \quad \text{or} \quad 0 = 4 + \frac{W}{1000} - 20 + \frac{W}{1500},$$

which is

$$16 = \frac{5W}{3000} \quad \text{or} \quad W = 9600$$

and $M = 30{,}000 - 9600 = 20{,}400$. Just to be safe, I want to compute the two marginal revenues at this allocation:

$$\mathrm{MR}_W = 20 - \frac{9600}{1000} = 10.40 \quad \text{and} \quad \mathrm{MR}_M = 24 - \frac{20{,}400}{1500} = 24 - 13.6 = 10.40$$

(I run this check both to make sure I didn't make an error in my algebra and for a reason that will become clear in part b.)

(b) With 60,000 seats to allocate, we do the same thing, but with 60,000 instead of 30,000. The key equation is

$$20 - \frac{W}{1000} = 24 - \frac{60{,}000 - W}{1500} \quad \text{or} \quad 0 = 4 + \frac{W}{1000} - 40 + \frac{W}{1500},$$

which gives us $36 = 5W/3000$ or $W = 21{,}600$ (and $M = 38{,}400$). Compute marginal revenues to be sure:

$$\mathrm{MR}_W = 20 - \frac{21{,}600}{1000} = -1.60 \quad \text{and} \quad \mathrm{MR}_M = 24 - \frac{38{,}400}{1500} = -1.60$$

Marginal revenues are equal. But they are both negative. This is a case where the constraint $W + M = 60{,}000$ doesn't bind: You want to allocate seats to the point where the two marginal revenues are both zero, which is $W = 20{,}000$ and $M = 36{,}000$. Any seats above 56,000 should be left empty.

Please note: At $W = 20{,}000$ and $M = 36{,}000$, the price per ticket for the Wolverton folks is 10. And there are empty seats. But to sell another seat to the Wolverton folks means you have to lower the price to all of them, and the lost revenue from those 20,000 fans is more than the gain in revenue from selling to the 20,001st. Also note that the price for Manteca fans is 12 (when $W = 36{,}000$). So Manteca fans will want to buy the cheapter tickets intended for Wolverton supporters. This scheme works only if (a) you can keep the two groups separate or (b) Manteca fans fear that buying tickets in the Wolverton-supporter section will entail unwanted medical bills. (This last comment applies as well to part a.)

(c) Now the marginal revenue for Manteca seats is the constant 12. So we want to give seats to Wolverton supporters to the point that $\mathrm{MR}_W = 12$, which is $20 - W/1000 = 12$, or $W = 8000$. Give Wolverton supporters 8000 seats, and give the rest (22,000) to Manteca supporters.

6.7 (a) I plotted the two functions (so this is not a rough drawing) in Excel, getting the picture shown in Figure S6.1. What is important here is the shape: Because there is a fixed cost, AC will be very high for low quantities. Marginal cost is linear and rising, so eventually AC will fall to the level of MC, and it will (thereafter) rise, with MC always (past this point) above AC.

Efficient scale is where they cross. The computer told me that this was at 400,000, but I could have found this algebraically, by equating AC and MC:

$$\frac{10{,}000{,}000}{x} + 50 + \frac{x}{16{,}000} = 50 + \frac{2x}{16{,}000} \quad \text{or} \quad 10{,}000{,}000 \cdot 16{,}000 = x^2,$$

which is $x = 400{,}000$. Minimum AC is MC at this level, which is $50 + 800{,}000/16{,}000 = 100$ (which I was able to read off the graph.)

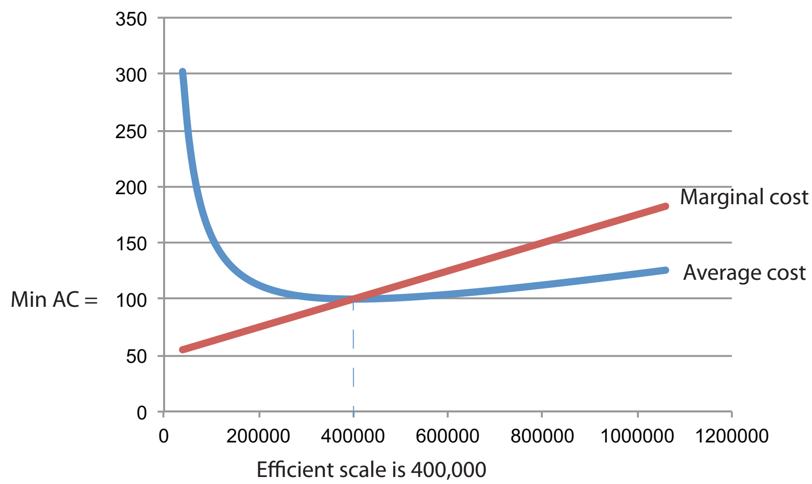

Figure S6.1. Solution to part a of Problem 6.7.

I'll answer part b both algebraically and, using Excel, graphically. To do this graphically, I need to add to Figure S6.1 the average and marginal revenue functions. See Figure S6.2. Profit is positive where AC is less than AR, which seems to be from around 60,000 units to 590,000. Profit is maximized where MC = MR, which is around 325,000.

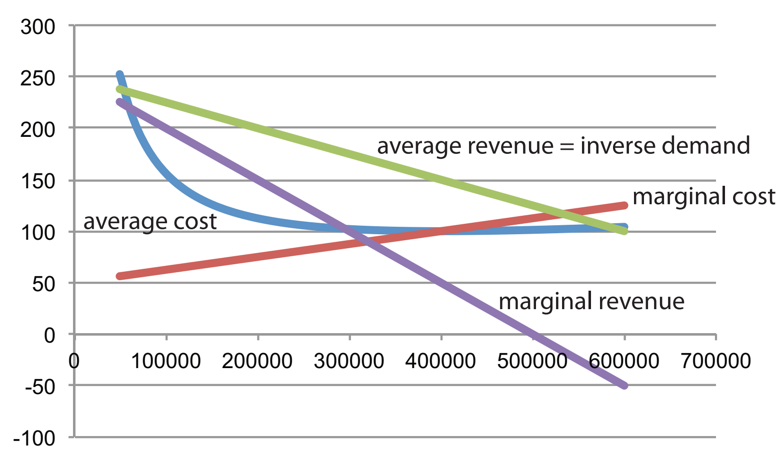

Figure S6.2. Solution to part b of Problem 6.7.

Now to do these things algebraically. AC is less than AR where

$$\frac{10{,}000{,}000}{x} + 50 + \frac{x}{16{,}000} < 250 - \frac{x}{4000}.$$

Multiply through by x, and the inequality holds where

$$250x - \frac{x^2}{4000} - \frac{x^2}{16{,}000} - 50x - 10{,}000{,}000 = 200x - \frac{5x^2}{16{,}000} - 10{,}000{,}000 > 0.$$

This is a parabola with a negative coefficient on the x^2 term, so it is positive between its two roots; to find the two roots, we use the quadratic formula: The roots are

$$\frac{-200 \pm \sqrt{200^2 - 4 \cdot (5/16{,}000)(10{,}000{,}000)}}{2 \cdot (-5/16{,}000)}.$$

If you carry out the computations, you get 54,670.017 (approximately) for the lower root and 585,329.98 (approximately) for the upper root. So between those two values of x, profit is positive.

And profit is maximized where MC = MR, which is where $250 - x/2000 = 50 + x/8000$, or $200 = 5x/8000$, or $x = 320{,}000$.

6.8 (a) Positive fixed cost, and rising marginal cost: Figure 6.3.

(b) No fixed cost, constant marginal cost: None of these. AC = MC = the constant function $AC(x) = MC(x) = 60$.

(c) Positive fixed cost and constant marginal cost: Figure 6.4(a).

(d) No fixed cost and rising (linear) marginal cost: Figure 6.4(b).

6.9 I'll use Excel: See Figure S6.3.

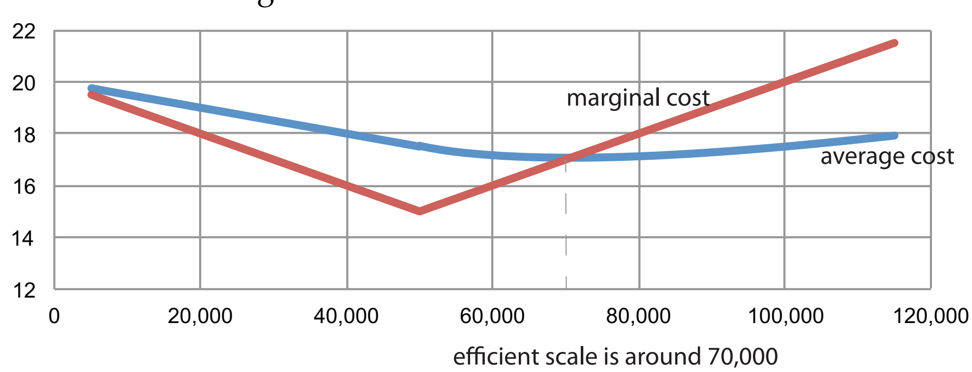

Figure S6.3. Solution to Problem 6.9.

The marginal cost function is piecewise linear, but with a kink at $x = 50{,}000$. So average cost is linear and decreasing to that point. Then it becomes nonlinear but continues to decrease until marginal cost cuts through it, when it begins to rise. It looks like this happens at $x = 70{,}000$, but to check, we need to see where AC = MC:

$$10 + \frac{x}{20{,}000} + \frac{250{,}000}{x} = 10 + \frac{2x}{20{,}000} \quad \text{or} \quad \frac{250{,}000}{x} = \frac{x}{20{,}000},$$

which is $x^2 = 20{,}000 \cdot 250{,}000 = 5{,}000{,}000{,}000$ or $x = \sqrt{5{,}000{,}000{,}000} = 70{,}710.68$, roughly.

6.10 The pictures are all in Figure S6.4. In each panel, the dashed line marks the profit-maximizing quantity, which is where MC = MR, of course. Panels d, e, and f answer part d of the question.

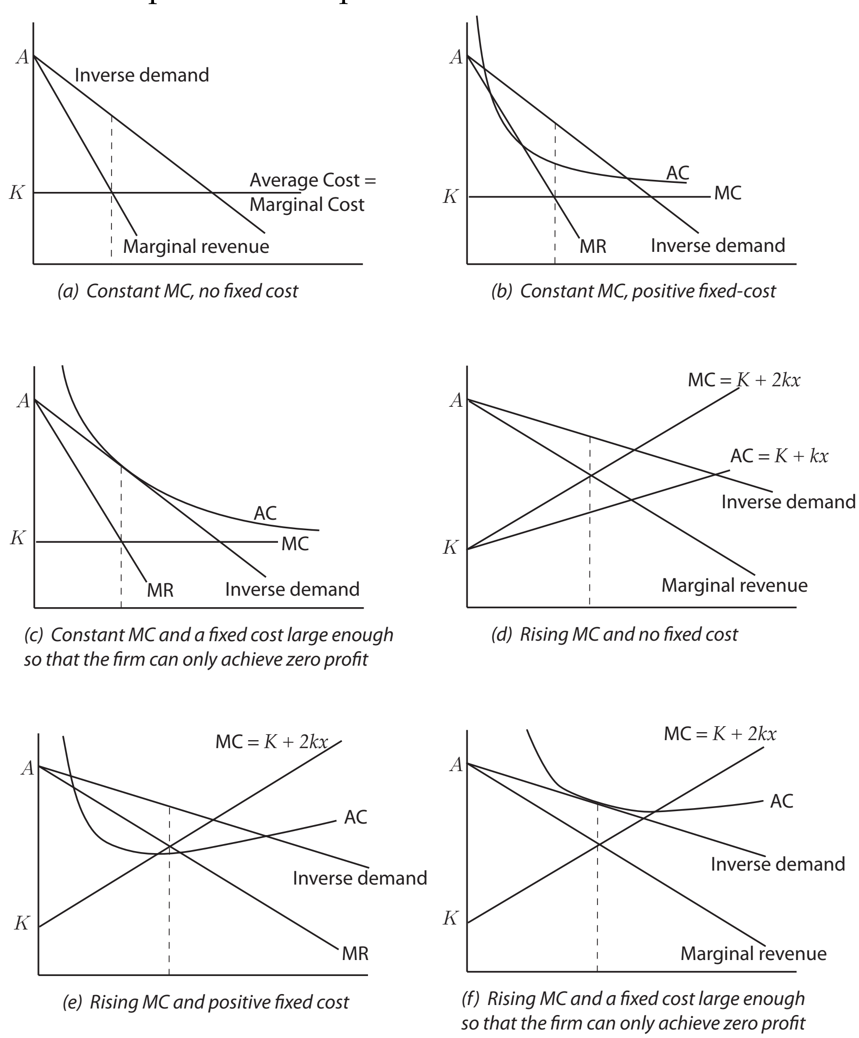

Figure S6.4. Solution to Problem 6.10.

6.11 No, this doesn't solve the problem. A product that makes a positive net contribution (before allocating fixed costs) might do so on the basis of very high sales figures, so this method of cost allocation would require the product to carry a very large proportion of the allocated costs, which would make it seem unprofitable.

6.12 This is a classic example of thinking about averages when you should think about margins. \$10 million is, presumably, the price per luxury box. Build another box, and you might (might!!!) have to lower the prices of the first 25. Suppose, for instance, to sell 26 boxes, the price per box had to be lowered to \$9.65 million. Then the 26th box would have a "marginal revenue" (in quotes because this should really be called incremental revenue) of

$$\$9.65 \text{ million per box } \times 26 \text{ boxes } - \$10 \text{ million per box } \times 25 \text{ boxes },$$

which is \$900k, not enough in additional revenues to cover the \$2 million construction cost.

Please note that this is a *might be true*. It is also possible that you can sell boxes for different prices, based on their proximity to the 50 yard line, and so this 26th box might not necessitate decreasing the cost of the first 25. But, in that case, the fact that you can sell the first 25 for an average of \$10 million doesn't imply that the 26th can be sold for just as much.

6.13 No, it is not good advice. The ROI calculation is an average calculation. You should make your decisions based on the marginal return you get for the marginal dollar spent on various keywords.