Review Problems II

The problems to follow provide you with the opportunity to review material covered in Part II of the book. Solutions to these problems are provided after all the problem statements.

II.1 A firm has total cost function $\text{TC}(x) = 10{,}000 + 10x$, and faces demand function $D(p) = 1000(50 - p)$. What is its profit-maximizing price, quantity, and profit level?

II.2 (a) Consider a firm that sells its product in two markets. Inverse demand in the first market is given by $P_1(x_1) = 20 - x_1/1000$, while inverse demand in the second market is given by $P_2(x_2) = 30 - x_2/400$. The firm can charge different prices in the two different markets. Its total cost depends on its total production level, $X = x_1 + x_2$, and is given by $\text{TC}(X) = 1000 + 10X + X^2/1000$. What production plan and prices in the two markets maximize this firm's profit? What is its (maximized) profit level?

(b) When the firm has optimized, it will charge different prices in the two markets and, in fact, P_2 will exceed P_1. What can you tell me (without doing any calculations with numbers) about the elasticities of demands that it will face in the two markets?

II.3 (a) A firm manufacturers two products, moiuyts and noiuyts. If it produces and sells x_m moiuyts and x_n noiuyts, the prices that pertain for the two goods are $p_m = 100 - x_m/100 - x_n/200$ and $p_n = 150 - x_n/100$, respectively. That is, the price of noiuyts depends only on the number of noiuyts produced and sold, but more noiuyts depresses the price of moiuyts. The marginal cost of a moiuyt is 20, and the marginal cost of a noiuyt is 25. What is the profit-maximizing production plan (prices and profit level) for this firm?

(b) Suppose that each moiuyt or noiuyt produced requires an oiuyt frame. The marginal cost of an oiuyt frame is 10, and the firm from part a felt it could procure all the oiuyt frames it wanted at this marginal cost. But it turns out that the firm can procure only 6000 oiuyt frames. Note well, if an oiuyt frame is used to make a moiuyt, there is an additional marginal cost of

10 (in addition to the cost 10 of the oiuyt frame), while if it is used to make a noiuyt, the additional (marginal) cost is 15. With the additional constraint imposed by the unavailibility of oiuyt frames, what is the profit-maximizing plan for the firm?

II.4 The total-cost function $\mathrm{TC}(x) = x^3 + 4x^2 + 10x + 99$ gives rise to a U-shaped average-cost function, where average costs first fall and then rise. Is the production level $x = 3$ above or below efficient scale? Justify your answer. (Hint: You *do not* need to find the level of efficient scale to answer this question.)

II.5 Consider the total-cost function $\mathrm{TC}(x) = 5000 + 2x^2$.

(a) What is the marginal-cost function?

(b) What is the average-cost function?

(c) Which value of x is the efficient scale of production, and what is the (minimum) average cost at the efficient scale?

II.6 Consider the demand function $D(p) = 1000p^{-2.5}$. What is the inverse demand function that goes with this demand function? What is the elasticity of demand at the price $p = 10$? What is the elasticity of demand that goes with the quantity $x = 32{,}000$? What is the marginal revenue function that goes with this demand function? (Can you answer the last question without computing total revenue and taking a derivative? How?)

II.7 (a) Demand for a particular product (sold by a single firm) has elasticity -2.5 at the price \$5.00, at which price demand equals 50,000 units. If the firm wishes to sell 52,500 units, approximately what price must be charged?

(b) This firm faces falling marginal revenue and rising marginal cost. Its marginal cost at the production level of 50,000 units is \$4.00 per unit. Is the profit-maximizing production quantity for this firm more than, less than, or precisely equal to 50,000 units?

(c) In particular, suppose that marginal costs and marginal revenues are close to constant over the range from 49,900 units to 50,100 units. What will be the impact on the profit of this firm if it produces 50,010 units instead of 50,000? (Note, I am not saying that this firm, if it maximizes its profit, would choose to produce at 50,000 units. You discovered whether it would or not in part b of this problem.)

II.8 A profit-maximizing firm produces 10,000 units per month and sells them for \$10.50 each. This firm's marginal cost at 10,000 units is \$4.20. What is the firm's elasticity of demand at the price \$10.50?

II.9 A profit-maximizing firm charges \$20 for its good, at which price it sells 50,000 units per year. It estimates that, if it raised its price by \$0.10, it would decrease demand by 750 units. What is its marginal cost of production at 50,000 units?

II.10 A firm sells its product to both commercial and industrial clients. Demand by industrial clients is $D_I(p) = 3000(550 - p)$ for $p \leq \$550$. (Industrial demand is 0 for prices above \$550.) Demand by commercial clients is $D_C(p) = 1000(750 - p)$ for $p \leq \$750$. (Commercial demand is 0 for prices above \$750.) Marginal costs are \$100 per unit.

(a) What single price maximizes profit, taking both groups into account? What is the firm's profit at that price?

(b) If the firm can set separate prices for the two types of customers, what pair of prices maximizes the firm's profit? How much profit (in total) does the firm make?

II.11 A price-setting firm located in the Republic of Freedonia has been producing 20,000 units of its one product each month, selling those units for \$20 each, with all sales taking place in Freedonia City. Its marginal cost of production at 20,000 units is \$12; it is convinced that the \$20 price is profit-maximizing for sales in Freedonia City.

(a) What is the elasticity of demand facing this firm in Freedonia City at the price \$20?

(b) The firm has received permission to sell its wares in Sylvanton (the capital of Sylvania, a neighboring Kingdom), although it must pay a \$1.50 per unit sold tariff to the Sylvanian government, which, combined with marginal costs of shipping, raises its marginal costs for each sale in Sylvanton by \$2.00 Accordingly, it started selling in Sylvanton for \$30 per unit, at which price it sold 3000 units per month. To gauge the market in Sylvanton, it ran a \$1 off sale (selling for \$29 per unit, a 3.33% reduction in price), and it found that its sales during the sales period rose from 3000 to 3200.

After this test-of-the-market at Sylvanton, the firm returned to a price of \$30 per unit in Sylvanton, selling 3000 units per month there, while selling 20,000 per month in Freedonia City at \$20 apiece.

It also found that the increase in production level to 23,000 caused its marginal cost of production to rise a bit, to \$12.25 per unit.

Based on all these numbers, and from a starting point of charging \$30 per unit in Sylvanton and \$20 per unit in Freedonia City, which of the following will increase the firm's level of profit:

a. Raise prices from current levels in both locations.
b. Lower prices from current levels in both locations.
c. Raise price in Freedonia City and lower price in Sylvanton.
d. Lower price in Freedonia City and raise price in Sylvanton.

Justify your answer.

II.12 Suppose that a vendor of cotton candy faces the marginal revenue function shown in Figure II.1. The discontinuities are due to kinks in the inverse demand curve. This vendor has the following total cost function: For levels of production between zero and 200 sticks, $\mathrm{TC}(C) = C + 50$. For levels of production above 200 sticks, $\mathrm{TC}(C) = 1.5C - 50$. What is the profit-maximizing level of production for the vendor, given this total cost function (and the marginal revenue function shown in Figure II.1)? (As a challenge, if I tell you that the inverse demand curve is continuous, can you reconstruct it from Figure II.1?)

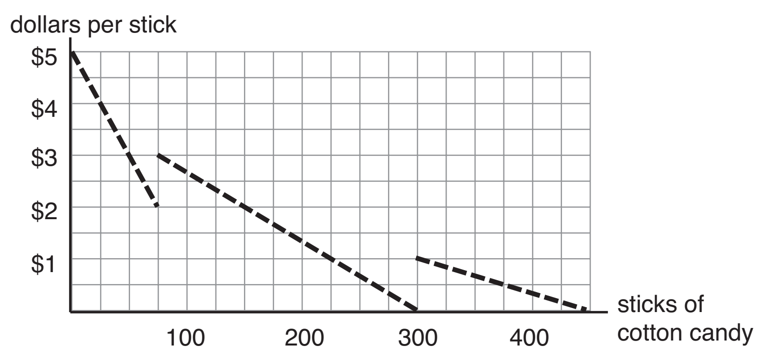

Figure II.1. Problem II.11: A marginal revenue function.

Solution to Problem II.1

The marginal cost is 10. To find marginal revenue, first find inverse demand: $x = D(p) = 1000(50 - p)$, so $x/1000 = 50 - p$ so

$$P(x) = 50 - \frac{x}{1000},$$

then total revenue and marginal revenue:

$$\mathrm{TR}(x) = 50x - \frac{x^2}{1000} \quad \text{so} \quad \mathrm{MR}(x) = 50 - \frac{2x}{1000}.$$

Equate marginal cost and marginal revenue:

$$50 - \frac{2x}{1000} = 10 \quad \text{or} \quad 40 = \frac{2x}{1000} \quad \text{or} \quad x = 20{,}000.$$

Plug this back into the inverse demand function to find the price $P(20{,}000) = 50 - 20{,}000/1000 = \$30$, and use the formulas for total revenue and total cost to find that profit equals \$390,000.

Solution to Problem II.2

(a) Marginal revenue in the first market is $\mathrm{MR}_1(x_1) = 20 - 2x_1/1000$, and marginal revenue in the second market is $\mathrm{MR}_2(x_2) = 30 - 2x_2/400$. Marginal cost is $\mathrm{MC}(X) = 10 + 2X/1000 = 10 + 2(x_1 + x_2)/1000$. The solution is where marginal revenue in each market equals marginal cost (in each market):

$$20 - \frac{2x_1}{1000} = 30 - \frac{2x_2}{400} = 10 + \frac{2(x_1 + x_2)}{1000}.$$

If you solve these two equations in two unknowns, the answer is $x_1 = 1250$ and $x_2 = 2500$, which, using the inverse demand functions, gives prices $P_1 = \$18.75$ and $P_2 = \$23.75$, for a profit of \$30,250.

(b) Since the marginal cost of an extra unit of x_1 (output for market 1) equals the marginal cost of an extra unit of x_2 (because total cost is a function of $x_1 + x_2$), the marginal revenues in the two markets must be equal. But, since $\mathrm{MR} = P(1 + 1/\nu)$, this implies that the term $(1 + 1/\nu_1) > (1 + 1/\nu_2)$, therefore $1/\nu_1 > 1/\nu_2$. Hence, $\nu_2 > \nu_1$. (Since elasticities are negative, this means that market 2, the market charged the higher price, has less elastic demand than market 1).

Solution to Problem II.3

(a) Total revenue is

$$\mathrm{TR}(x_m, x_n) = 100x_m - \frac{x_m^2}{100} - \frac{x_m x_n}{200} + 150x_n - \frac{x_n^2}{100},$$

so marginal revenues in the two products are

$$\mathrm{MR}_m(x_m, x_n) = 100 - \frac{2x_m}{100} - \frac{x_n}{200} \quad \text{and} \quad \mathrm{MR}_n(x_m, x_n) = 150 - \frac{x_m}{200} - \frac{2x_n}{100}.$$

We have to solve for x_m and x_n, where we equate the marginal revenue in moiuyts to their marginal cost of 20 and the marginal revenue in noiuyts to their marginal cost of 25. This gives $x_m = 2600$ and $x_n = 5600$, for prices $p_m = \$46$ and $p_n = \$94$ and profit equal to \$454,000.

(b) The solution to this problem requires that you equate the marginal profits for the two products. Marginal profit for moiuyts is the marginal revenue for moiuyts less their \$10 variable marginal cost:

$$\mathrm{M}\pi_m(x_m, x_n) = 90 - \frac{2x_m}{100} - \frac{x_n}{200},$$

while marginal profit for noiuyts is the marginal revenue for noiuyts less their \$15 variable marginal cost:

$$135 - \frac{x_m}{200} - \frac{2x_n}{100}.$$

You need not subtract off the marginal cost of the oiuyt frames, since that would be the same in the two, so would cancel out when you equate them. This gives one equation; the second equation is that the total production run has to be 6000, or $x_m + x_n = 6000$. Solve these and you'll find $x_m = 1500$ and $x_n = 4500$, hence $p_m = \$62.50, p_n = \$105$, and the profit equals \$423,750. Note that this gives marginal profit figures of \$37.50, more than the \$10 marginal cost of an oiuyt frame, so the firm certainly wishes to use all 6000.

Solution to Problem II.4

For the given total cost function, marginal cost is $\mathrm{MC}(x) = 3x^2 + 8x + 10$, while average cost is $\mathrm{AC}(x) = x^2 + 4x + 10 + 99/x$. Therefore, at $x = 3$, $\mathrm{MC}(3) = 27 + 24 + 10 = 61$, while $\mathrm{AC}(3) = 9 + 12 + 10 + 33 = 64$. Hence, at $x = 3$, average cost exceeds marginal cost and average cost is falling. Since we know average cost is U-shaped, this means that $x = 3$ is below the efficient scale.

Solution to Problem II.5

(a) $\mathrm{MC}(x) = 4x$

(b) $\mathrm{AC}(x) = 5000/x + 2x$

(c) To find the efficient scale, you can either (1) take the derivative of average cost and set it equal to 0, or (2) look for where marginal cost equals average cost. Following plan (1), we get

$$-1 \times \frac{5000}{x^2} + 2 = 0 \quad \text{or} \quad 2 = \frac{5000}{x^2} \quad \text{or} \quad 5000 = 2x^2.$$

Following plan (2), we get

$$\frac{5000}{x} + 2x = 4x \quad \text{or} \quad \frac{5000}{x} = 2x \quad \text{or} \quad 5000 = 2x^2.$$

(Of course, they lead to the same equation.) This gives $x^2 = 2500 \quad \text{or} \quad x = 50$, at which point, average cost = marginal cost = $4 \times 50 = 200$.

Solution to Problem II.6

To find the inverse-demand function, invert the relationship $x = 1000p^{-2.5}$ as follows:

$$\frac{x}{1000} = p^{-2.5} \quad \text{so} \quad \left(\frac{x}{1000}\right)^{-1/2.5} = p,$$

and the inverse-demand function is therefore

$$P(x) = \left(\frac{x}{1000}\right)^{-0.4}.$$

Rather than find the elasticity at a price of \$10 or at a quantity of 32,000, let me compute the entire elasticity function:

$$\nu(p) = D'(p) \times \frac{p}{D(p)} = -2500p^{-3.5} \times \frac{p}{1000p^{-2.5}} = -2.5.$$

That is, the elasticity of demand is -2.5 at all prices and so at all quantities.

This makes it relatively easy to give the marginal-revenue function:

$$\mathrm{MR}(x) = P(x)\left(1 + \frac{1}{\hat{\nu}(x)}\right) = \left(\frac{x}{1000}\right)^{-0.4}\left(1 + \frac{1}{-2.5}\right) = 0.6\left(\frac{x}{1000}\right)^{-0.4}.$$

Solution to Problem II.7

(a) An increase of 2,500 units on a base of 50,000 is an increase of 5%. If elasticity is -2.5, this means a 2% decrease in price. On a base of \$5.00, a 2% decrease is \$0.10, so price must fall to \$4.90.

(b) By the formula, MR(50,000) = $\$5(1 - \frac{1}{2.5}) = \$3$. Since marginal cost is \$4, 50,000 is a point at which marginal costs exceed marginal revenues. Since marginal cost rises and marginal revenue falls, to maximize profit, this firm must decrease its level of production.

(c) In particular, an increase in production by 10 units means (approximately) a rise in revenues by \$30 and a rise in costs of \$40, for a net decrease in profit of \$10.

Solution to Problem II.8

Because the firm maximizes its profit at 10,000 units per month, its marginal cost equals its marginal revenue at that level of production. Since its marginal cost is \$4.20, this is also its marginal revenue. But marginal revenue is $P(1 + 1/\hat{\nu})$, therefore

$$4.2 = 10.5\left(1 + \frac{1}{\hat{\nu}(10{,}000)}\right).$$

The solution to this equation is $\hat{\nu}(10{,}000) = \nu(\$10.50) = -1.6667$.

Solution to Problem II.9

A \$0.10 raise in price is a 0.5% raise on a base of \$20. A decrease of 750 in demand on a base of 50,000 is a 1.5% fall. Hence, the elasticity of demand at this price–quantity pair is -3. Therefore, the firm's marginal revenue at 50,000 units is $20(1 + \frac{1}{-3}) = \frac{40}{3} = \$13.33$. Since this firm maximizes its profit at this level of production, this is also its marginal cost.

Solution to Problem II.10

(a) Demand at prices below \$550 is the sum of the two pieces of demand, or $3000(550 - p) + 1000(750 - p) = 2{,}400{,}000 - 4000p$. Inverse demand for these prices (corresponding to a quantity of 200,000 or more) is $P(x) = 600 - x/4000$. Therefore, marginal revenue over this range is $\mathrm{MR}(x) = 600 -$

$x/2000$. Equate this to marginal cost, and you get

$$600 - x/2000 = 100 \quad \text{or} \quad x = 1{,}000{,}000.$$

This corresponds to a price of \$350, and thus a profit of $(350-100)(1{,}000{,}000) =$ \$250 million.

To be sure this is the profit-maximizing production plan, we check to see whether marginal cost ever equals marginal revenue for prices above \$550 and quantities below 200,000. But, at $x = 200{,}000$, marginal revenue (in the first segment, for prices between \$750 and \$550) has fallen to \$350, still well above the \$100 marginal cost.

(b) For commercial customers, set a price of \$425, selling 325,000 units, for a profit of \$105,625,000. For industrial customers, set a price of \$325, selling 675,000 units, for a profit of \$151,875,000. With the ability to discriminate between the two groups, the firm makes \$257.5 million, a 3% gain over the nondiscrimination solution.

Solution to Problem II.11

(a) Use the formula $\text{MR} = p(1 + 1/\hat{\nu})$ and the profit-maximizing condition MC = MR. Mrginal cost at 20,000 units is \$12, and since this is profit maximizing, it is also MR. Hence

$$12 = 20\left(1+\frac{1}{\hat{\nu}}\right) \quad \text{or} \quad \frac{12}{20} = 1+\frac{1}{\hat{\nu}} \quad \text{or} \quad 0.4 = \frac{1}{\hat{\nu}} \quad \text{which gives} \quad \hat{\nu}(20,000) = -2.5.$$

And since the quantity 20,000 is the same point on the demand function as is the price \$12, this means that $\nu(\$12) = -2.5$.

(b) A 3.33% fall in price in Sylvanton gave a $200/3000 = 0.0667$ or 6.667% increase in quantity, so the elasticity of demand in Sylvanton around \$30 per unit there is -2. Hence, using the formula again, marginal revenue in Sylvanton is

$$\$30\left(1 + \frac{1}{-2}\right) = \$15.$$

Hence, at a \$20 in Freedonia City, MR = \$12 while MC = \$12.25—the firm wants to sell fewer units in Freedonia city, which means raising the price there. And at \$30 apiece in Sylvanton, MR is \$15, while MC is $12.25 + 2 =$

$14.25, so the firm will raise profit by selling more in Sylvanton, which means lowering the price there.

Hence the correct answer is c: Raise the price in Freedonia City and lower the price in Sylvanton.

Solution to Problem II.12

In Figure I.2 (overleaf), we superimpose on the picture of marginal revenue the marginal cost function, which is $\mathrm{MC}(c) = 1$ for $c < 200$ and $\mathrm{MC}(c) = 1.5$ for $c > 200$. In this case, marginal revenue exceeds marginal cost for all levels of production below 200 and marginal cost exceeds marginal revenue for all levels above 200 units, so the profit-maximizing point will be 200 units.

As for the challenge, I graph the corresponding continuous inverse demand function in Figure I.3 as a dashed line. I will let you figure out how I got it with the following substantial hint: For each of the line segments in inverse demand and for each of the line segments in marginal revenue, using a ruler carefully extend the segment "back" to the quantity 0. (This is already done for the first segments; you must do it for the other two.)

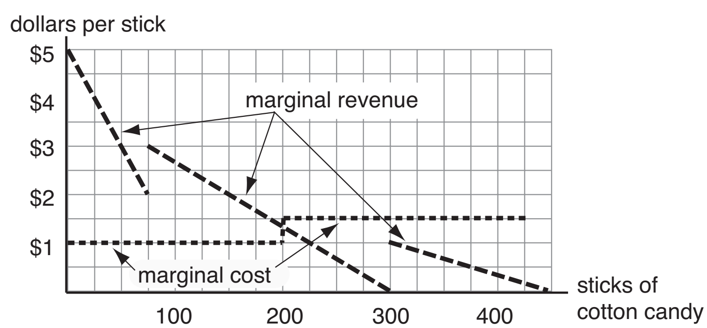

Figure II.2. Problem I.11: Marginal revenue and marginal cost.

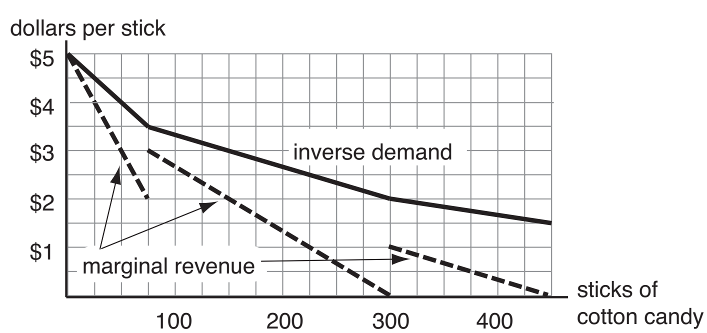

Figure II.3. Problem I.11 challenge: The inverse demand function.